CHAPTER

7 Iterative Methods for Solving Linear Systems

7.1 Introduction

The previous chapter considered the approximation of the solution of a linear system using *direct* methods, techniques that would produce the exact solution if all the calculations were performed using exact arithmetic. In this chapter we describe some popular *iterative* techniques, which require an initial approximation to the solution. These methods are not expected to return the exact solution even if all the calculations could be performed using exact arithmetic. In many instances, however, they are more effective than the direct methods because they can require far less computational effort and round-off error is reduced. This is particularly true when the matrix is **sparse**—that is, when it has a high percentage of zero entries.

Some additional material from linear algebra is needed to describe the convergence of the iterative methods. Principally, we need to have a measure of how close two vectors are to one another because the object of an iterative method is to determine an approximation that is within a certain tolerance of the exact solution.

In Section 7.2, the notion of a norm is used to show how various forms of distance between vectors can be described. We will also see how this concept can be extended to describe the norm of—and, consequently, the distance between—matrices. In Section 7.3, matrix eigenvalues and eigenvectors are described, and we consider the connection between these concepts and the convergence of an iterative method.

Section 7.4 describes the elementary Jacobi and Gauss-Seidel iterative methods. By analyzing the size of the largest eigenvalue of a matrix associated with an iterative method, we can determine conditions that predict the likelihood of convergence of the method. In Section 7.5 we introduce the SOR method. This is a commonly applied iterative technique because it reduces the approximation errors faster than the Jacobi and Gauss-Seidel methods.

In Section 7.6 we discuss some of the concerns that should be addressed when applying either an iterative or a direct technique for approximating the solution to a linear system.

The conjugate gradient method is presented in Section 7.7. This method, with preconditioning, is the technique most often used for sparse, positive-definite matrices.

7.2 Convergence of Vectors

The distance between the real numbers x and y is $|x - y|$. In Chapter 2 we saw that the stopping techniques for the iterative root-finding techniques used this measure to estimate the accuracy of approximate solutions and to determine when an approximation was sufficiently accurate. The iterative methods for solving systems of equations use similar logic, so the first step is to determine a way to measure the distance between n-dimensional vectors because this is the form that is taken by the solution to a system of equations.

Vector Norms

Let $\mathbb{R}^n$ denote the set of all n-dimensional column vectors with real number coefficients. It is a space-saving convenience to use the transpose notation presented in Section 6.4 when such a vector is represented in terms of its components. Generally, we write the vector

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \text{as} \quad \mathbf{x} = (x_1, x_2, \ldots, x_n)^t.$$

Vector Norm on $\mathbb{R}^n$

A vector norm on $\mathbb{R}^n$ is a function, $\|\cdot\|$, from $\mathbb{R}^n$ into $\mathbb{R}$ with the following properties:

(i) $\|\mathbf{x}\| \geq 0$ for all $\mathbf{x} \in \mathbb{R}^n$,

(ii) $\|\mathbf{x}\| = 0$ if and only if $\mathbf{x} = (0, 0, \ldots, 0)^t \equiv \mathbf{0}$,

(iii) $\|\alpha\mathbf{x}\| = |\alpha|\|\mathbf{x}\|$ for all $\alpha \in \mathbb{R}$ and $\mathbf{x} \in \mathbb{R}^n$,

(iv) $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$.

For our purposes, we need only two specific norms on $\mathbb{R}^n$. (A third is presented in Exercise 2.)

The l_2 and l_∞ norms for the vector $\mathbf{x} = (x_1, x_2, \ldots, x_n)^t$ are defined by

$$\|\mathbf{x}\|_2 = \left\{\sum_{i=1}^{n} x_i^2\right\}^{1/2} \quad \text{and} \quad \|\mathbf{x}\|_\infty = \max_{1 \leq i \leq n} |x_i|.$$

The l_2 norm is called the **Euclidean norm** of the vector $\mathbf{x}$ because it represents the usual notion of distance from the origin when $\mathbf{x}$ is in $\mathbb{R}^1 \equiv \mathbb{R}$, $\mathbb{R}^2$, or $\mathbb{R}^3$. For example, the l_2 norm of the vector $\mathbf{x} = (x_1, x_2, x_3)^t$ gives the length of the straight line joining the points $(0, 0, 0)$ and (x_1, x_2, x_3); that is, the length of the shortest path between those two points. Figure 7.1 shows the boundary of those vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$ that have l_2 norm less than 1. Figure 7.2 gives a similar illustration for the l_∞ norm.

Figure 7.1

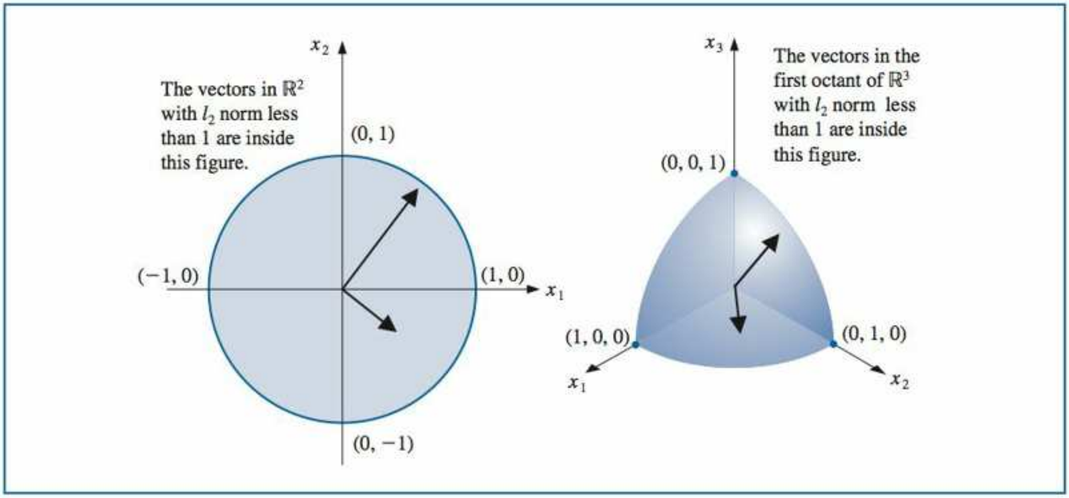
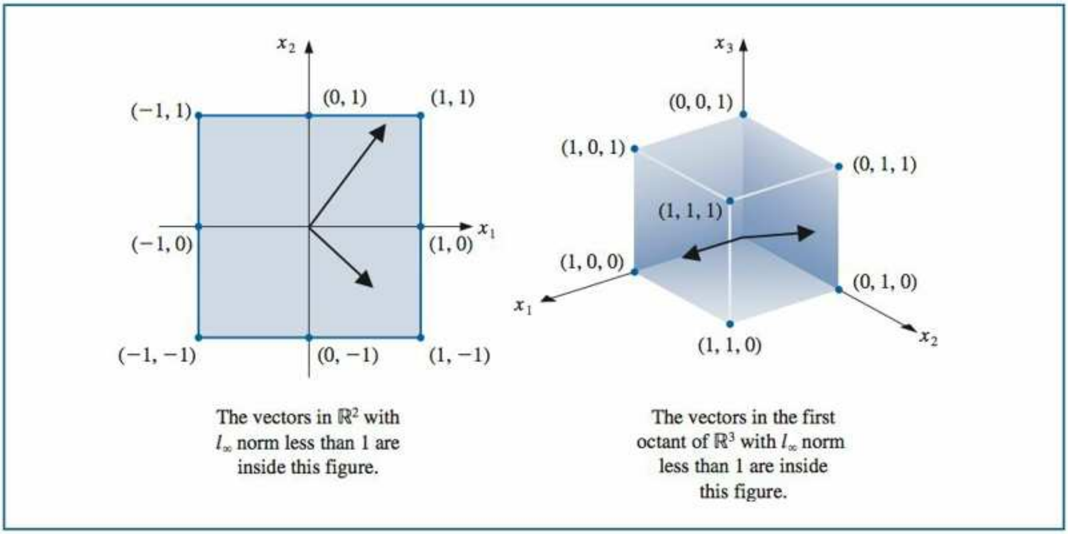


Figure 7.2



Example 1 Determine the l_2 norm and the l_∞ norm of the vector $\mathbf{x} = (-1, 1, -2)^t$.

Solution The vector $\mathbf{x} = (-1, 1, -2)^t$ in $\mathbb{R}^3$ has norms

$$\|\mathbf{x}\|_2 = \sqrt{(-1)^2 + (1)^2 + (-2)^2} = \sqrt{6}$$

and

$$\|\mathbf{x}\|_\infty = \max\{|-1|, |1|, |-2|\} = 2.$$

Notice that $\|\mathbf{x}\|_\infty < \|\mathbf{x}\|_2$ in this example. ■

There are many forms of this inequality, hence many discoverers. Augustin Louis Cauchy (1789–1857) describes this inequality in 1821 in *Cours d'Analyse Algébrique*, the first rigorous calculus book. An integral form of the equality appears in the work of Viktor Yakovlevich Bunyakovsky (1804–1889) in 1859, and Hermann Amandus Schwarz (1843–1921) used a double integral form of this inequality in 1885. More details on the history can be found in [Stee].

Showing that $\|\mathbf{x}\|_\infty = \max_{1\le i\le n} |x_i|$ satisfies the conditions necessary for a norm on $\mathbb{R}^n$ follows directly from the truth of similar statements concerning absolute values of real numbers. In the case of the l_2 norm, it is also easy to demonstrate the first three of the required properties, but the fourth,

$$\|\mathbf{x}+\mathbf{y}\|_2 \le \|\mathbf{x}\|_2 + \|\mathbf{y}\|_2,$$

is more difficult to show. To demonstrate this inequality we need the Cauchy-Buniakowsky-Schwarz inequality, which states that for any $\mathbf{x} = (x_1, x_2, \dots, x_n)^t$ and $\mathbf{y} = (y_1, y_2, \dots, y_n)^t$,

$$\sum_{i=1}^{n} |x_i y_i| \le \left\{\sum_{i=1}^{n} x_i^2\right\}^{1/2} \left\{\sum_{i=1}^{n} y_i^2\right\}^{1/2}. \tag{7.1}$$

With this it follows that $\|\mathbf{x}+\mathbf{y}\|_2 \le \|\mathbf{x}\|_2 + \|\mathbf{y}\|_2$ because

$$\begin{aligned}\|\mathbf{x}+\mathbf{y}\|_2^2 &= \sum_{i=1}^{n} x_i^2 + 2\sum_{i=1}^{n} x_i y_i + \sum_{i=1}^{n} y_i^2 \le \sum_{i=1}^{n} x_i^2 + 2\sum_{i=1}^{n} |x_i y_i| + \sum_{i=1}^{n} y_i^2 \\ &\le \sum_{i=1}^{n} x_i^2 + 2\left\{\sum_{i=1}^{n} x_i^2\right\}^{1/2} \left\{\sum_{i=1}^{n} y_i^2\right\}^{1/2} + \sum_{i=1}^{n} y_i^2 = (\|\mathbf{x}\|_2 + \|\mathbf{y}\|_2)^2.\end{aligned}$$

Distance between Vectors in $\mathbb{R}^n$

The norm of a vector gives a measure for the distance between the vector and the origin, so the distance between two vectors is the norm of the difference of the vectors.

Distance between Vectors

If $\mathbf{x} = (x_1, x_2, \dots, x_n)^t$ and $\mathbf{y} = (y_1, y_2, \dots, y_n)^t$ are vectors in $\mathbb{R}^n$, the l_2 and l_∞ distances between $\mathbf{x}$ and $\mathbf{y}$ are defined by

$$\|\mathbf{x}-\mathbf{y}\|_2 = \left\{\sum_{i=1}^{n} (x_i - y_i)^2\right\}^{1/2} \quad \text{and} \quad \|\mathbf{x}-\mathbf{y}\|_\infty = \max_{1\le i\le n} |x_i - y_i|.$$

Example 2 The linear system

$$\begin{aligned}3.3330x_1 + 15920x_2 - 10.333x_3 &= 15913,\\ 2.2220x_1 + 16.710x_2 + 9.6120x_3 &= 28.544,\\ 1.5611x_1 + 5.1791x_2 + 1.6852x_3 &= 8.4254\end{aligned}$$

has the exact solution $\mathbf{x} = (x_1, x_2, x_3)^t = (1, 1, 1)^t$, and Gaussian elimination performed using five-digit rounding arithmetic and partial pivoting produces the approximate solution

$$\tilde{\mathbf{x}} = (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3)^t = (1.2001, 0.99991, 0.92538)^t.$$

Determine the l_2 and l_∞ distances between the exact and approximate solutions.

Solution Measurements of $\mathbf{x} - \tilde{\mathbf{x}}$ are given by

$$\begin{aligned}\|\mathbf{x} - \tilde{\mathbf{x}}\|_2 &= \left[(1 - 1.2001)^2 + (1 - 0.99991)^2 + (1 - 0.92538)^2\right]^{1/2} \\ &= [(0.2001)^2 + (0.00009)^2 + (0.07462)^2]^{1/2} = 0.21356.\end{aligned}$$

and

$$\begin{aligned}\|\mathbf{x} - \tilde{\mathbf{x}}\|_\infty &= \max\{|1 - 1.2001|, |1 - 0.99991|, |1 - 0.92538|\} \\ &= \max\{0.2001, 0.00009, 0.07462\} = 0.2001\end{aligned}$$

Although the components $\tilde{x}_2$ and $\tilde{x}_3$ are good approximations to x_2 and x_3, the component $\tilde{x}_1$ is a poor approximation to x_1, and $|x_1 - \tilde{x}_1|$ dominates both norms. ■

The distance concept in $\mathbb{R}^n$ is used to define the limit of a sequence of vectors. A sequence $\{\mathbf{x}^{(k)}\}_{k=1}^{\infty}$ of vectors in $\mathbb{R}^n$ is said to **converge** to $\mathbf{x}$ with respect to the norm $\|\cdot\|$ if, given any $\varepsilon > 0$, there exists an integer $N(\varepsilon)$ such that

$$\|\mathbf{x}^{(k)} - \mathbf{x}\| < \varepsilon \quad \text{for all } k \geq N(\varepsilon).$$

Example 3 Show that

$$\mathbf{x}^{(k)} = \left(x_1^{(k)}, x_2^{(k)}, x_3^{(k)}, x_4^{(k)}\right)^t = \left(1, 2 + \frac{1}{k}, \frac{3}{k^2}, e^{-k}\sin k\right)^t$$

converges to $\mathbf{x} = (1, 2, 0, 0)^t$ with respect to the l_∞ norm.

Solution Let $\varepsilon > 0$ be given. For each of the component functions,

$$\begin{aligned}
\lim_{k\to\infty} 1 = 1, &\quad \text{so an integer } N_1(\varepsilon) \text{ exists with } \left|x_1^{(k)} - 1\right| < \varepsilon \quad \text{for all } k \geq N_1(\varepsilon),\\
\lim_{k\to\infty} (2 + 1/k) = 2, &\quad \text{so an integer } N_2(\varepsilon) \text{ exists with } \left|x_2^{(k)} - 2\right| < \varepsilon \quad \text{for all } k \geq N_2(\varepsilon),\\
\lim_{k\to\infty} 3/k^2 = 0, &\quad \text{so an integer } N_3(\varepsilon) \text{ exists with } \left|x_3^{(k)} - 0\right| < \varepsilon \quad \text{for all } k \geq N_3(\varepsilon),\\
\lim_{k\to\infty} e^{-k}\sin k = 0, &\quad \text{so an integer } N_4(\varepsilon) \text{ exists with } \left|x_4^{(k)} - 0\right| < \varepsilon \quad \text{for all } k \geq N_4(\varepsilon).
\end{aligned}$$

Let

$$N(\varepsilon) = \max\{N_1(\varepsilon), N_2(\varepsilon), N_3(\varepsilon), N_4(\varepsilon)\}.$$

Then when $k \geq N(\varepsilon)$, we have

$$||\mathbf{x}^{(k)} - \mathbf{x}||_\infty = \max\left\{\left|x_1^{(k)} - 1\right|, \left|x_2^{(k)} - 2\right|, \left|x_3^{(k)} - 0\right|, \left|x_4^{(k)} - 0\right|\right\} < \varepsilon,$$

so $\mathbf{x}^{(k)}$ converges to $\mathbf{x}$. ■

In Example 3 we implicitly used the fact that a sequence of vectors $\{\mathbf{x}^{(k)}\}_{k=1}^{\infty}$ converges in the norm $\|\cdot\|_\infty$ to the vector $\mathbf{x}$ if and only if, for each $i = 1, 2, \ldots, n$, the sequence $\{x_i^{(k)}\}_{k=1}^{\infty}$ converges to x_i, the ith component of $\mathbf{x}$. This makes the determination of convergence for the norm $\|\cdot\|_\infty$ relatively easy.

To show directly that the sequence in Example 3 converges to $(1, 2, 0, 0)^t$ with respect to the l_2 norm is quite complicated. However, suppose that $\mathbf{x}$ is a vector in $\mathbb{R}^n$ and j is an index with the property that

$$\|\mathbf{x}\|_\infty = \max_{i=1,\ldots,n} |x_i| = |x_j|.$$

Then

$$\|\mathbf{x}\|_\infty^2 = |x_j|^2 = x_j^2 \le \sum_{i=1}^n x_i^2 = \|\mathbf{x}\|_2^2 \quad \text{and} \quad \|\mathbf{x}\|_2^2 = \sum_{i=1}^n x_i^2 \le \sum_{i=1}^n x_j^2 = nx_j^2 = n\|\mathbf{x}\|_\infty^2.$$

This gives the norm inequalities

$$\|\mathbf{x}\|_\infty \le \|\mathbf{x}\|_2 \le \sqrt{n}\|\mathbf{x}\|_\infty.$$

This implies that the sequence of vectors $\{\mathbf{x}^{(k)}\}$ also converges to $\mathbf{x}$ in $\mathbb{R}^n$ with respect to $\|\cdot\|_2$ if and only if $\lim_{k\to\infty} x_i^{(k)} = x_i$ for each $i = 1, 2, \ldots, n$, since this is when the sequence converges in the l_∞ norm.

In fact, it can be shown that all norms on $\mathbb{R}^n$ are equivalent with respect to convergence; that is,

- if $\|\cdot\|$ and $\|\cdot\|'$ are any two norms on $\mathbb{R}^n$ and $\{\mathbf{x}^{(k)}\}_{k=1}^\infty$ has the limit $\mathbf{x}$ with respect to $\|\cdot\|$, then $\{\mathbf{x}^{(k)}\}_{k=1}^\infty$ has the limit $\mathbf{x}$ with respect to $\|\cdot\|'$.

Since a vector sequence converges in the l_∞ norm precisely when each of its component sequences converges, we have the following.

Vector Sequence Convergence

The following statements are equivalent:

(i) The sequence of vectors $\{\mathbf{x}^{(k)}\}$ converges to $\mathbf{x}$ in some norm.

(ii) The sequence of vectors $\{\mathbf{x}^{(k)}\}$ converges to $\mathbf{x}$ in every norm.

(iii) For each of the component functions $x_i^{(k)}$ of $\mathbf{x}^{(k)}$, we have $\lim_{k\to\infty} x_i^{(k)} = x_i$.

Matrix Norms and Distances

In the subsequent sections, we will need methods for determining the distance between $n \times n$ matrices. This again requires the use of a norm.

Matrix Norm

A matrix norm on the set of all $n \times n$ matrices is a real-valued function, $\|\cdot\|$, defined on this set, satisfying for all $n \times n$ matrices A and B and all real numbers α:

(i) $\|A\| \ge 0$,

(ii) $\|A\| = 0$, if and only if A is O, the matrix with all zero entries,

(iii) $\|\alpha A\| = |\alpha|\|A\|$,

(iv) $\|A + B\| \le \|A\| + \|B\|$,

(v) $\|AB\| \le \|A\|\|B\|$.

Every vector norm produces an associated natural matrix norm.

A **distance between $n \times n$ matrices** A and B with respect to this matrix norm is $\|A - B\|$. Although matrix norms can be obtained in various ways, the only norms we consider are those that are natural consequences of a vector norm.

Natural Matrix Norm

If $\|\cdot\|$ is a vector norm on $\mathbb{R}^n$, the natural matrix norm on the set of $n \times n$ matrices given by $\|\cdot\|$ is defined by

$$\|A\| = \max_{\|\mathbf{x}\|=1} \|A\mathbf{x}\|.$$

So, the l_2 and l_∞ matrix norms are, respectively,

$$\|A\|_2 = \max_{\|\mathbf{x}\|_2=1} \|A\mathbf{x}\|_2 \quad \text{(the } l_2 \text{ norm)} \quad \text{and} \quad \|A\|_\infty = \max_{\|\mathbf{x}\|_\infty=1} \|A\mathbf{x}\|_\infty \quad \text{(the } l_\infty \text{ norm)}.$$

When $n = 2$ these norms have the geometric representations shown in Figures 7.3 and 7.4.

Figure 7.3

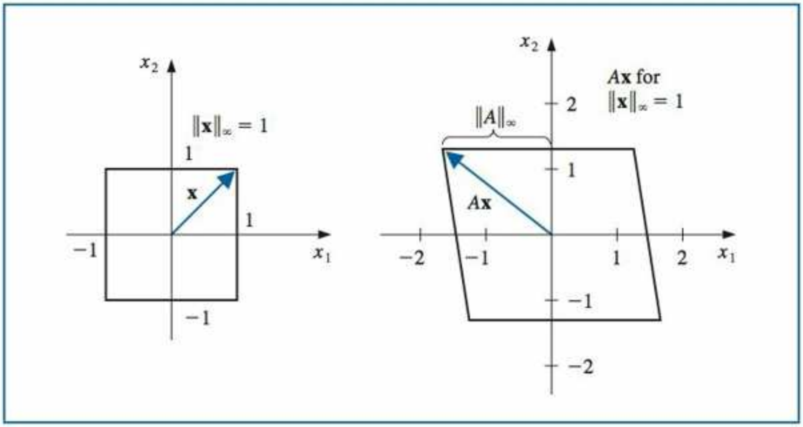
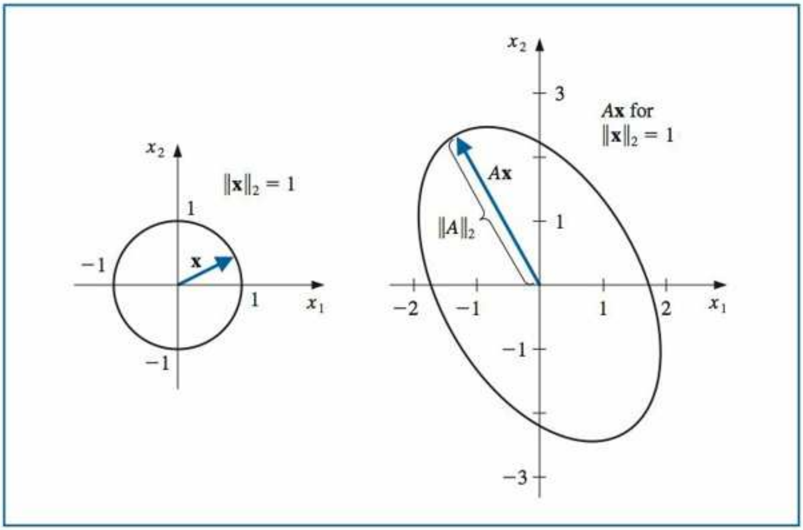


Figure 7.4



The l_∞ norm of a matrix has a representation with respect to the entries of the matrix that makes it particularly easy to compute.

l_∞ Matrix Norm Characterization

$$\|A\|_\infty = \max_{1\le i\le n} \sum_{j=1}^{n} |a_{ij}|.$$

Example 4 Determine $\|A\|_\infty$ for the matrix

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & -1 \\ 5 & -1 & 1 \end{bmatrix}.$$

Solution We have

$$\sum_{j=1}^{3} |a_{1j}| = |1| + |2| + |-1| = 4, \quad \sum_{j=1}^{3} |a_{2j}| = |0| + |3| + |-1| = 4,$$

and

$$\sum_{j=1}^{3} |a_{3j}| = |5| + |-1| + |1| = 7.$$

So $\|A\|_\infty = \max\{4, 4, 7\} = 7$. ▪

The l_2 norm of a matrix is not as easily determined, but in the next section we will discover an alternative method for finding this norm.

EXERCISE SET 7.2

1. Find $\|\mathbf{x}\|_\infty$ and $\|\mathbf{x}\|_2$ for the following vectors.

a. $\mathbf{x} = \left(3, -4, 0, \frac{3}{2}\right)^t$

b. $\mathbf{x} = (2, 1, -3, 4)^t$

c. $\mathbf{x} = (\sin k, \cos k, 2^k)^t$ for a fixed positive integer k

d. $\mathbf{x} = (4/(k+1), 2/k^2, k^2 e^{-k})^t$ for a fixed positive integer k

2. **a.** Verify that $\|\cdot\|_1$ is a norm for $\mathbb{R}^n$ (called the l_1 norm), where

$$\|\mathbf{x}\|_1 = \sum_{i=1}^{n} |x_i|.$$

b. Find $\|\mathbf{x}\|_1$ for the vectors given in Exercise 1.

3. Show that the following sequences are convergent, and find their limits.

a. $\mathbf{x}^{(k)} = (1/k, e^{1-k}, -2/k^2)^t$

b. $\mathbf{x}^{(k)} = (e^{-k}\cos k, k\sin(1/k), 3 + k^{-2})^t$

c. $\mathbf{x}^{(k)} = \left(ke^{-k^2}, (\cos k)/k, \sqrt{k^2+k} - k\right)^t$

d. $\mathbf{x}^{(k)} = (e^{1/k}, (k^2+1)/(1-k^2), (1/k^2)(1+3+5+\cdots+(2k-1)))^t$

4. Find $\|\cdot\|_\infty$ for the following matrices.

a. $\begin{bmatrix} 10 & 15 \\ 0 & 1 \end{bmatrix}$ **b.** $\begin{bmatrix} 10 & 0 \\ 15 & 1 \end{bmatrix}$

c. $\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$ **d.** $\begin{bmatrix} 4 & -1 & 7 \\ -1 & 4 & 0 \\ -7 & 0 & 4 \end{bmatrix}$

5. The following linear systems $A\mathbf{x} = \mathbf{b}$ have $\mathbf{x}$ as the actual solution and $\tilde{\mathbf{x}}$ as an approximate solution. Compute $\|\mathbf{x} - \tilde{\mathbf{x}}\|_\infty$ and $\|A\tilde{\mathbf{x}} - \mathbf{b}\|_\infty$.

a. $\frac{1}{2}x_1 + \frac{1}{3}x_2 = \frac{1}{63},$
$\frac{1}{3}x_1 + \frac{1}{4}x_2 = \frac{1}{168},$
$\mathbf{x} = \left(\frac{1}{7}, -\frac{1}{6}\right)^t,$
$\tilde{\mathbf{x}} = (0.142, -0.166)^t.$

b. $x_1 + 2x_2 + 3x_3 = 1,$
$2x_1 + 3x_2 + 4x_3 = -1,$
$3x_1 + 4x_2 + 6x_3 = 2,$
$\mathbf{x} = (0, -7, 5)^t,$
$\tilde{\mathbf{x}} = (-0.33, -7.9, 5.8)^t.$

c. $x_1 + 2x_2 + 3x_3 = 1,$
$2x_1 + 3x_2 + 4x_3 = -1,$
$3x_1 + 4x_2 + 6x_3 = 2,$
$\mathbf{x} = (0, -7, 5)^t,$
$\tilde{\mathbf{x}} = (-0.2, -7.5, 5.4)^t.$

d. $0.04x_1 + 0.01x_2 - 0.01x_3 = 0.06,$
$0.2x_1 + 0.5x_2 - 0.2x_3 = 0.3,$
$x_1 + 2x_2 + 4x_3 = 11,$
$\mathbf{x} = (1.827586, 0.6551724, 1.965517)^t,$
$\tilde{\mathbf{x}} = (1.8, 0.64, 1.9)^t.$

6. The l_1 matrix norm, defined by $\|A\|_1 = \max_{\|\mathbf{x}\|_1=1} \|A\mathbf{x}\|_1$, can be computed using the formula

$$\|A\|_1 = \max_{1\le j\le n} \sum_{i=1}^{n} |a_{ij}|,$$

where the l_1 vector norm is defined in Exercise 2. Find the l_1 norm of the matrices in Exercise 4.

7. Show by example that $\|\cdot\|_{\otimes}$, defined by $\|A\|_{\otimes} = \max_{1\le i,j\le n} |a_{ij}|$, does not define a matrix norm.

8. Show that $\|\cdot\|_{\textcircled{1}}$, defined by

$$\|A\|_{\textcircled{1}} = \sum_{i=1}^{n}\sum_{j=1}^{n} |a_{ij}|,$$

is a matrix norm. Find $\|\cdot\|_{\textcircled{1}}$ for the matrices in Exercise 4.

9. Show that if $\|\cdot\|$ is a vector norm on $\mathbb{R}^n$, then $\|A\| = \max_{\|\mathbf{x}\|=1} \|A\mathbf{x}\|$ is a matrix norm.

7.3 Eigenvalues and Eigenvectors

An $n \times m$ matrix can be considered as a function that uses matrix multiplication to take m-dimensional vectors into n-dimensional vectors. So an $n \times n$ matrix A takes the set of

n-dimensional vectors into itself. In this case certain nonzero vectors can have $\mathbf{x}$ and $A\mathbf{x}$ parallel, which means that a constant λ exists with $A\mathbf{x} = \lambda\mathbf{x}$, or that $(A - \lambda I)\mathbf{x} = \mathbf{0}$. There is a close connection between these numbers λ and the likelihood that an iterative method based on A will converge. We will consider the connection in this section.

For a square $n \times n$ matrix A, the **characteristic polynomial** of A is defined by

$$p(\lambda) = \det(A - \lambda I).$$

Because of the way the determinant of a matrix is defined, p is an nth-degree polynomial and, consequently, has at most n distinct zeros, some of which might be complex. These zeros of p are called the **eigenvalues** of the matrix A.

The result on page 256 in Chapter 6, then, implies that the following are equivalent:

- λ is an eigenvalue of A,
- $A - \lambda I$ does not have an inverse,
- there exists a vector $\mathbf{x} \neq \mathbf{0}$ with $A\mathbf{x} = \lambda\mathbf{x}$,
- $\det(A - \lambda I) = 0$.

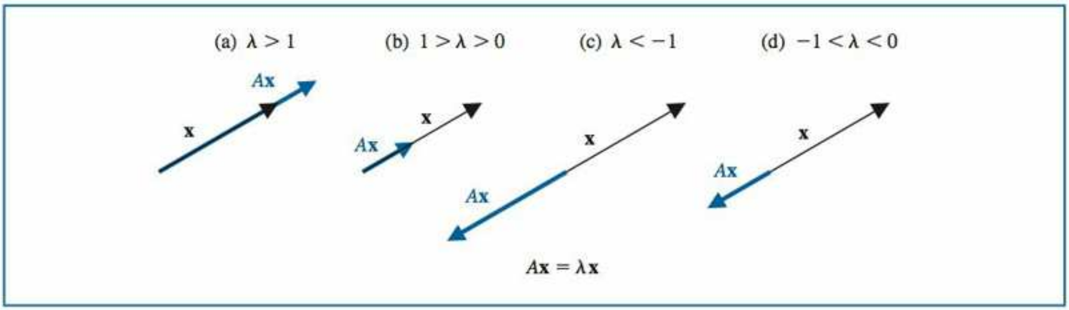
If $\mathbf{x}$ is a nonzero vector with $A\mathbf{x} = \lambda\mathbf{x}$, then $\mathbf{x}$ is called an **eigenvector** of A corresponding to the eigenvalue λ. Note that if $\mathbf{x}$ is an eigenvector of A corresponding to the eigenvalue λ, then any nonzero scalar multiple $\alpha\mathbf{x}$ of $\mathbf{x}$ is also an eigenvector of A corresponding to λ because

$$A(\alpha\mathbf{x}) = \alpha(A\mathbf{x}) = \alpha(\lambda\mathbf{x}) = \lambda(\alpha\mathbf{x}).$$

The prefix *eigen* comes from the German adjective meaning "to own" and is synonymous in English with the word *characteristic*. Each matrix has its own eigen- or characteristic equation, with corresponding eigen- or characteristic values and functions.

If $\mathbf{x}$ is an eigenvector associated with the eigenvalue λ, then $A\mathbf{x} = \lambda\mathbf{x}$, so the matrix A takes the vector $\mathbf{x}$ into a scalar multiple of itself. When λ is a real number and $\lambda > 1$, A has the effect of stretching $\mathbf{x}$ by a factor of λ. When $0 < \lambda < 1$, A shrinks $\mathbf{x}$ by a factor of λ. When $\lambda < 0$, the effects are similar, but the direction is reversed (see Figure 7.5).

Figure 7.5



Example 1 Determine the eigenvalues and corresponding eigenvectors for the matrix

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 2 \\ 1 & -1 & 4 \end{bmatrix}.$$

Solution The characteristic polynomial of A is

$$p(\lambda) = \det(A - \lambda I) = \det \begin{bmatrix} 2-\lambda & 0 & 0 \\ 1 & 1-\lambda & 2 \\ 1 & -1 & 4-\lambda \end{bmatrix}$$
$$= -(\lambda^3 - 7\lambda^2 + 16\lambda - 12) = -(\lambda - 3)(\lambda - 2)^2,$$

so there are two eigenvalues of A: $\lambda_1 = 3$ and $\lambda_2 = 2$.

An eigenvector $\mathbf{x}_1 \neq \mathbf{0}$ corresponding to the eigenvalue $\lambda_1 = 3$ is a solution to the vector-matrix equation $(A - 3 \cdot I)\mathbf{x}_1 = \mathbf{0}$, so

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 1 & -2 & 2 \\ 1 & -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_1 \\ x_1 - 2x_2 + 2x_3 \\ x_1 - x_2 + x_3 \end{bmatrix},$$

which implies that $x_1 = 0$ and $x_2 = x_3$. Any nonzero value of x_3 produces an eigenvector for the eigenvalue $\lambda_1 = 3$. For example, when $x_3 = 1$ we have the eigenvector $\mathbf{x}_1 = (0, 1, 1)^t$. Any eigenvector of A corresponding to $\lambda = 3$ is a nonzero multiple of $\mathbf{x}_1$.

An eigenvector $\mathbf{x}_2 \neq \mathbf{0}$ of A associated with the eigenvalue $\lambda_2 = 2$ is a solution of the system $(A - 2I)\mathbf{x} = \mathbf{0}$, so

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & -1 & 2 \\ 1 & -1 & 2 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ x_1 - x_2 + 2x_3 \\ x_1 - x_2 + 2x_3 \end{bmatrix}.$$

In this case the eigenvector has only to satisfy the equation

$$x_1 - x_2 + 2x_3 = 0,$$

which can be done in various ways. For example, when $x_1 = 0$ we have $x_2 = 2x_3$, so one choice would be $\mathbf{x}_2 = (0, 2, 1)^t$. We could also choose $x_2 = 0$, which requires that $x_1 = -2x_3$. Hence $\mathbf{x}_3 = (-2, 0, 1)^t$ gives a second eigenvector for the eigenvalue $\lambda_2 = 2$, one that is not a multiple of $\mathbf{x}_2$.

The eigenvectors of A corresponding to the eigenvalue $\lambda_2 = 2$ generate an entire plane. This plane is described by all vectors of the form

$$\alpha\mathbf{x}_2 + \beta\mathbf{x}_3 = (-2\beta, 2\alpha, \alpha + \beta)^t,$$

for arbitrary constants α and β, provided that at least one of the constants is nonzero. ■

The next example illustrates that even some very simple matrices can have no real eigenvalues.

Example 2 Show that there are no nonzero vectors $\mathbf{x}$ in $\mathbb{R}^2$ with $B\mathbf{x}$ parallel to $\mathbf{x}$ if

$$B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Solution The eigenvalues of B are the solutions to the characteristic polynomial

$$0 = \det(B - \lambda I) = \det \begin{bmatrix} -\lambda & 1 \\ -1 & -\lambda \end{bmatrix} = \lambda^2 + 1,$$

so the eigenvalues of B are the complex numbers $\lambda_1 = i$ and $\lambda_2 = -i$. A corresponding eigenvector $\mathbf{x}$ for λ_1 needs to satisfy

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -ix_1 + x_2 \\ -x_1 - ix_2 \end{bmatrix},$$

that is, $0 = -ix_1 + x_2$, so $x_2 = ix_1$, and an eigenvector for $\lambda_1 = i$ is $(1, i)^t$. In a similar manner, an eigenvector for $\lambda_2 = -i$ is $(1, -i)^t$.

If $\mathbf{x}$ is an eigenvector of B, then exactly one of its components is real and the other is complex. As a consequence, there is no real constant λ and nonzero vector $\mathbf{x}$ in $\mathbb{R}^2$ with $B\mathbf{x} = \lambda\mathbf{x}$, and hence there is no nonzero vector $\mathbf{x}$ in $\mathbb{R}^2$ with $B\mathbf{x}$ parallel to $\mathbf{x}$. ■

MATLAB provides methods to directly compute the eigenvalues and eigenvectors of a matrix. We first define the matrix A by

```
A = [1 0 2; 0 1 -1; -1 1 1]
```

The characteristic polynomial is determined with

```
p=poly(A)
```

giving

$$p = \;1.0000\;\; -3.0000\;\; 6.0000\;\; -4.0000$$

The numbers are the coefficients of the characteristic polynomial in descending order, so

$$p(\lambda) = \lambda^3 - 3\lambda^2 + 6\lambda - 4.$$

We can now compute the roots of the polynomial to obtain the eigenvalues with

```
roots(p)
```

The most direct way to obtain eigenvalues is with the `eig` command.

```
eig(A)
```

If we want the corresponding eigenvectors, we enter `eig` as

```
[V, D] = eig(A)
```

which produces the following matrix V and vector D. We have rounded the entries in V so that it will display on one line.

$$\begin{array}{llll} V = & -0.70710678 & -0.70710678 & 0.70710678 \\ & 0.35355339 + 0.00000000i & 0.35355339 - 0.00000000i & 0.70710678 \\ & -0.00000000 - 0.61237244i & -0.00000000 + 0.61237244i & 0.00000000 \end{array}$$

$$\begin{array}{ll} D = & 0.999999999999999 + 1.732050807568876i \\ & 0.999999999999999 - 1.732050807568876i \\ & 1.000000000000000 \end{array}$$

The columns of V are eigenvectors of A corresponding to the eigenvalues in the rows of D.

The notions of eigenvalues and eigenvectors are introduced here for a specific computational convenience, but these concepts arise frequently in the study of physical systems. In fact, they are of sufficient interest that most of Chapter 9 is devoted to their approximation.

Spectral Radius

The **spectral radius** $\rho(A)$ of a matrix A is defined by

$$\rho(A) = \max |\lambda|, \quad \text{where } \lambda \text{ is an eigenvalue of } A.$$

(*Note*: For complex $\lambda = \alpha + \beta i$, we have $|\lambda| = (\alpha^2 + \beta^2)^{1/2}$.)

Example 3 Determine the spectral radius of the matrices

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 2 \\ 1 & -1 & 4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Solution In Example 1 we found that the eigenvalues of A were $\lambda_1 = 3$ and $\lambda_2 = 2$. So

$$\rho(A) = \max\{|3|, |2|\} = 3,$$

and in Example 2 we found that the eigenvalues of B were $\lambda_1 = i$ and $\lambda_2 = -i$. So

$$\rho(B) = \max\{\sqrt{1^2}, \sqrt{(-1)^2}\} = 1.$$

The spectral radius is closely related to the norm of a matrix.

l_2 Matrix Norm Characterization

If A is an $n \times n$ matrix, then

(i) $\|A\|_2 = [\rho(A^t A)]^{1/2}$;

(ii) $\rho(A) \leq \|A\|$ for any natural norm.

The first part of this result is the computational method for determining the l_2 norm of matrices that we mentioned at the end of the previous section.

Example 4 Determine the l_2 norm of

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ -1 & 1 & 2 \end{bmatrix}.$$

Solution To apply part (i) of the l_2 Matrix Norm Characterization, we need to calculate $\rho(A^t A)$, so we need the eigenvalues of $A^t A$.

$$A^t A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ -1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 2 & -1 \\ 2 & 6 & 4 \\ -1 & 4 & 5 \end{bmatrix}.$$

If

$$0 = \det(A^t A - \lambda I) = \det \begin{bmatrix} 3-\lambda & 2 & -1 \\ 2 & 6-\lambda & 4 \\ -1 & 4 & 5-\lambda \end{bmatrix}$$
$$= -\lambda^3 + 14\lambda^2 - 42\lambda = -\lambda(\lambda^2 - 14\lambda + 42),$$

then $\lambda = 0$ or $\lambda = 7 \pm \sqrt{7}$. So

$$||A||_2 = \sqrt{\rho(A^t A)} = \sqrt{\max\{0, 7-\sqrt{7}, 7+\sqrt{7}\}} = \sqrt{7+\sqrt{7}} \approx 3.106.$$

The operations in Example 4 can also be performed using MATLAB. First define

```
A = [1 1 0; 1 2 1; -1 1 2]
```

then compute its transpose and determine $A^t A$, and the eigenvalues of $A^t A$

```
u = eig(A'*A)
```

This gives the eigenvalues as

$$u = 0.000000000000003$$
$$4.354248688935409$$
$$9.645751311064592$$

The square root of the largest eigenvalue is the l_2 norm of A

```
sqrt(u(3))
```

which MATLAB gives as 3.105760987433610.

The l_2 norm of A can also be directly computed with

```
norm(A)
```

The l_∞ norm of A is found with `norm(A,Inf)`.

Convergent Matrices

In studying iterative matrix techniques, it is of particular importance to know when the powers of a matrix become small (that is, when all of the entries approach zero). We call an $n \times n$ matrix A **convergent** if, for each $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, n$, we have

$$\lim_{k\to\infty} (A^k)_{ij} = 0.$$

Example 5 Show that

$$A = \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{2} \end{bmatrix}$$

is a convergent matrix.

Solution Computing the powers of A, we obtain:

$$A^2 = \begin{bmatrix} \frac{1}{4} & 0 \\ \frac{1}{4} & \frac{1}{4} \end{bmatrix}, \quad A^3 = \begin{bmatrix} \frac{1}{8} & 0 \\ \frac{3}{16} & \frac{1}{8} \end{bmatrix}, \quad A^4 = \begin{bmatrix} \frac{1}{16} & 0 \\ \frac{1}{8} & \frac{1}{16} \end{bmatrix},$$

and, in general,

$$A^k = \begin{bmatrix} (\frac{1}{2})^k & 0 \\ \frac{k}{2^{k+1}} & (\frac{1}{2})^k \end{bmatrix}.$$

So A is a convergent matrix because

$$\lim_{k\to\infty} \left(\frac{1}{2}\right)^k = 0 \quad \text{and} \quad \lim_{k\to\infty} \frac{k}{2^{k+1}} = 0.$$

■

The following important connection exists between the spectral radius of a matrix and the convergence of the matrix.

Convergent Matrix Equivalences

The following are equivalent statements:

(i) A is a convergent matrix.

(ii) $\lim_{n\to\infty} \|A^n\| = 0$, for some natural norm.

(iii) $\lim_{n\to\infty} \|A^n\| = 0$, for all natural norms.

(iv) $\rho(A) < 1$.

(v) $\lim_{n\to\infty} A^n\mathbf{x} = \mathbf{0}$, for every $\mathbf{x}$.

EXERCISE SET 7.3

1. Compute the eigenvalues and associated eigenvectors of the following matrices.

a. $\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$

b. $\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$

c. $\begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix}$

d. $\begin{bmatrix} 1 & 1 \\ -2 & -2 \end{bmatrix}$

e. $\begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

f. $\begin{bmatrix} -1 & 2 & 0 \\ 0 & 3 & 4 \\ 0 & 0 & 7 \end{bmatrix}$

g. $\begin{bmatrix} 2 & 1 & 1 \\ 2 & 3 & 2 \\ 1 & 1 & 2 \end{bmatrix}$

h. $\begin{bmatrix} 3 & 2 & -1 \\ 1 & -2 & 3 \\ 2 & 0 & 4 \end{bmatrix}$

2. Find the spectral radius for each matrix in Exercise 1.

3. Show that

$$A_1 = \begin{bmatrix} 1 & 0 \\ \frac{1}{4} & \frac{1}{2} \end{bmatrix}$$

is not convergent, but

$$A_2 = \begin{bmatrix} \frac{1}{2} & 0 \\ 16 & \frac{1}{2} \end{bmatrix}$$

is convergent.

4. Which of the matrices in Exercise 1 are convergent?

5. Find the $\|\cdot\|_2$ norms of the matrices in Exercise 1.

6. Show that if λ is an eigenvalue of a matrix A and $\|\cdot\|$ is a vector norm, then an eigenvector $\mathbf{x}$ associated with λ exists with $\|\mathbf{x}\| = 1$.

7. Find 2×2 matrices A and B for which $\rho(A + B) > \rho(A) + \rho(B)$. (This shows that $\rho(A)$ cannot be a matrix norm.)

8. Show that if A is symmetric, then $\|A\|_2 = \rho(A)$.

9. Let λ be an eigenvalue of the $n \times n$ matrix A and $\mathbf{x} \neq \mathbf{0}$ be an associated eigenvector.

a. Show that λ is also an eigenvalue of A^t.

b. Show that for any integer $k \geq 1$, λ^k is an eigenvalue of A^k with eigenvector $\mathbf{x}$.

c. Show that if A^{-1} exists, then $1/\lambda$ is an eigenvalue of A^{-1} with eigenvector $\mathbf{x}$.

d. Let $\alpha \neq \lambda$ be given. Show that if $(A-\alpha I)^{-1}$ exists, then $1/(\lambda-\alpha)$ is an eigenvalue of $(A-\alpha I)^{-1}$ with eigenvector $\mathbf{x}$.

10. In Exercise 8 of Section 6.4, it was assumed that the contribution a female beetle of a certain type made to the future years' beetle population could be expressed in terms of the matrix

$$A = \begin{bmatrix} 0 & 0 & 6 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \end{bmatrix},$$

where the entry in the ith row and jth column represents the probabilistic contribution of a beetle of age j onto the next year's female population of age i.

a. Does the matrix A have any real eigenvalues? If so, determine them and any associated eigenvectors.

b. If a sample of this species was needed for laboratory test purposes that would have a constant proportion in each age group from year to year, what criteria could be imposed on the initial population to ensure that this requirement would be satisfied?

7.4 The Jacobi and Gauss-Seidel Methods

In this section we describe the elementary Jacobi and Gauss-Seidel iterative methods. These are classic methods that date to the late eighteenth century, but they find current application in problems where the matrix is large and has mostly zero entries in predictable locations. Applications of this type are common, for example, in the study of large integrated circuits and in the numerical solution of boundary-value problems and partial-differential equations.

General Iteration Methods

An iterative technique for solving the $n \times n$ linear system $A\mathbf{x} = \mathbf{b}$ starts with an initial approximation $\mathbf{x}^{(0)}$ to the solution $\mathbf{x}$ and generates a sequence of vectors $\{\mathbf{x}^{(k)}\}_{k=1}^{\infty}$ that converges to $\mathbf{x}$. These iterative techniques involve a process that converts the system $A\mathbf{x} = \mathbf{b}$ into an equivalent system of the form $\mathbf{x} = T\mathbf{x} + \mathbf{c}$ for some $n \times n$ matrix T and vector $\mathbf{c}$.

After the initial vector $\mathbf{x}^{(0)}$ is selected, the sequence of approximate solution vectors is generated by computing

$$\mathbf{x}^{(k)} = T\mathbf{x}^{(k-1)} + \mathbf{c}$$

for each $k = 1, 2, 3, \ldots$.

The following result provides an important connection between the eigenvalues of the matrix T and the expectation that the iterative method will converge.

Convergence and the Spectral Radius

The sequence

$$\mathbf{x}^{(k)} = T\mathbf{x}^{(k-1)} + \mathbf{c}$$

converges to the unique solution of $\mathbf{x} = T\mathbf{x} + \mathbf{c}$ for any $\mathbf{x}^{(0)}$ in $\mathbb{R}^n$ if and only if $\rho(T) < 1$.