CHAPTER

5 Numerical Solution of Initial-Value Problems

5.1 Introduction

Differential equations are used to model problems that involve the change of some variable with respect to another. These problems require the solution to an initial-value problem—that is, the solution to a differential equation that satisfies a given initial condition.

In many real-life situations, the differential equation that models the problem is too complicated to solve exactly, and one of two approaches is taken to approximate the solution. The first approach is to simplify the differential equation to one that can be solved exactly, and then use the solution of the simplified equation to approximate the solution to the original equation. The other approach, the one we examine in this chapter, involves finding methods for directly approximating the solution of the original problem. This is the approach commonly taken because more accurate results and realistic error information can be obtained.

The methods we consider in this chapter do not produce a continuous approximation to the solution of the initial-value problem. Rather, approximations are found at certain specified, and often equally-spaced, points. Some method of interpolation, commonly cubic Hermite, is used if intermediate values are needed.

The first part of the chapter concerns approximating the solution $y(t)$ to a problem of the form

$$\frac{dy}{dt} = f(t, y), \quad \text{for } a \leq t \leq b,$$

subject to an initial condition

$$y(a) = \alpha.$$

These techniques form the core of the study because more general procedures use these as a base. Later in the chapter we deal with the extension of these methods to a system of first-order differential equations in the form

$$\frac{dy_1}{dt} = f_1(t, y_1, y_2, \ldots, y_n),$$
$$\frac{dy_2}{dt} = f_2(t, y_1, y_2, \ldots, y_n),$$
$$\vdots$$
$$\frac{dy_n}{dt} = f_n(t, y_1, y_2, \ldots, y_n),$$

for $a \leq t \leq b$, subject to the initial conditions

$$y_1(a) = \alpha_1, \quad y_2(a) = \alpha_2, \quad \ldots, \quad y_n(a) = \alpha_n.$$

We also examine the relationship of a system of this type to the general nth-order initial-value problem of the form

$$y^{(n)} = f\left(t, y, y', y'', \ldots, y^{(n-1)}\right)$$

for $a \le t \le b$, subject to the multiple initial conditions

$$y(a) = \alpha_0, \quad y'(a) = \alpha_1, \ldots, y^{(n-1)}(a) = \alpha_{n-1}.$$

Well-Posed Problems

Before describing the methods for approximating the solution to our basic problem, we consider some situations that ensure the solution will exist. In fact, because we will not be solving the given problem, only an approximation to the problem, we need to know when problems that are close to the given problem have solutions that accurately approximate the solution to the given problem. This property of an initial-value problem is called **well-posed**, and these are the problems for which numerical methods are appropriate. The following result shows that the class of well-posed problems is quite broad.

Well-Posed Condition

Suppose that f and f_y, its first partial derivative with respect to y, are continuous for t in $[a, b]$ and for all y. Then the initial-value problem

$$y' = f(t, y), \quad \text{for } a \le t \le b, \quad \text{with } y(a) = \alpha,$$

has a unique solution $y(t)$ for $a \le t \le b$, and the problem is well-posed.

Example 1 Consider the initial-value problem

$$y' = 1 + t\sin(ty), \quad \text{for } 0 \le t \le 2, \quad \text{with } y(0) = 0.$$

Since the functions

$$f(t, y) = 1 + t\sin(ty) \qquad \text{and} \qquad f_y(t, y) = t^2\cos(ty)$$

are both continuous for $0 \le t \le 2$ and for all y, a unique solution exists to this well-posed initial-value problem.

If you have taken a course in differential equations, you might attempt to determine the solution to this problem by using one of the techniques you learned in that course. ■

5.2 Taylor Methods

The methods in this section use Taylor polynomials and the knowledge of the derivative at a node to approximate the value of the function at a new node.

Many of the numerical methods we saw in the first four chapters have an underlying derivation from Taylor's Theorem. The approximation of the solution to initial-value problems is no exception. In this case, the function we need to expand in a Taylor polynomial is the (unknown) solution to the problem, $y(t)$. In its most elementary form, this leads to **Euler's Method**. Although Euler's method is seldom used in practice, the simplicity of its derivation illustrates the technique used for more advanced procedures, without the cumbersome algebra that accompanies these constructions.

The use of elementary difference methods to approximate the solution to differential equations was one of the numerous mathematical topics that was first presented to the mathematical public by the most prolific of mathematicians, Leonhard Euler (1707–1783).

The objective of Euler's method is to find, for a given positive integer N, an approximation to the solution of a problem of the form

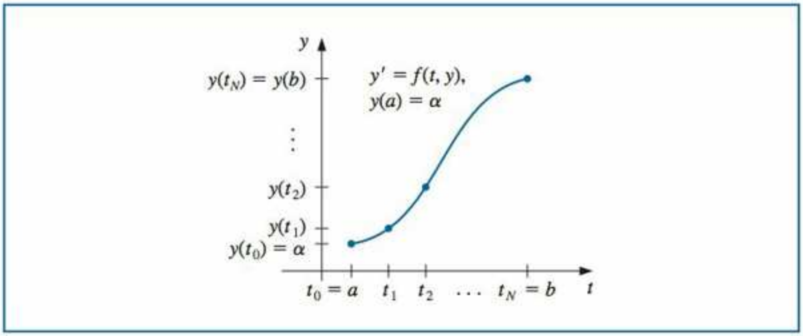
$$\frac{dy}{dt} = f(t, y), \quad \text{for } a \le t \le b, \quad \text{with } y(a) = \alpha$$

at the $N + 1$ equally-spaced **mesh points** $\{t_0, t_1, t_2, \ldots, t_N\}$ (see Figure 5.1). The common distance between the points, $h = (b - a)/N$, is called the **step size**, and

$$t_i = a + ih, \quad \text{for each } i = 0, 1, \ldots N.$$

Approximations at other values of t in $[a, b]$ can then be found using interpolation.

Figure 5.1



Suppose that $y(t)$, the solution to the problem, has two continuous derivatives on $[a, b]$, so that for each $i = 0, 1, 2, \ldots, N - 1$, Taylor's Theorem implies that

$$y(t_{i+1}) = y(t_i) + (t_{i+1} - t_i)y'(t_i) + \frac{(t_{i+1} - t_i)^2}{2} y''(\xi_i),$$

for some number ξ_i in (t_i, t_{i+1}). Letting $h = (b - a)/N = t_{i+1} - t_i$, we have

$$y(t_{i+1}) = y(t_i) + hy'(t_i) + \frac{h^2}{2} y''(\xi_i),$$

and, since $y(t)$ satisfies the differential equation $y'(t) = f(t, y(t))$,

$$y(t_{i+1}) = y(t_i) + hf(t_i, y(t_i)) + \frac{h^2}{2} y''(\xi_i).$$

Euler's method constructs the approximation w_i to $y(t_i)$ for each $i = 1, 2, \ldots, N$ by deleting the error term in this equation. This produces a *difference equation* that approximates the differential equation. The term **local error** refers to the error at the given step if it is assumed that all the previous results are exact. The true, or accumulated, error of the method is called **global error**.

Euler's Method

$$w_0 = \alpha,$$

$$w_{i+1} = w_i + hf(t_i, w_i),$$

where $i = 0, 1, \ldots, N - 1$, with local error $\frac{1}{2}y''(\xi_i)h^2$ for some ξ_i in (t_i, t_{i+1}).

Illustration In Example 1 we will use Euler's method to approximate the solution to

$$y' = y - t^2 + 1, \quad 0 \le t \le 2, \quad y(0) = 0.5,$$

at $t = 2$. Here we will simply illustrate the steps in the technique when we have $h = 0.5$.

For this problem $f(t, y) = y - t^2 + 1$, so

$$w_0 = y(0) = 0.5;$$
$$w_1 = w_0 + 0.5(w_0 - (0.0)^2 + 1) = 0.5 + 0.5(1.5) = 1.25;$$
$$w_2 = w_1 + 0.5(w_1 - (0.5)^2 + 1) = 1.25 + 0.5(2.0) = 2.25;$$
$$w_3 = w_2 + 0.5(w_2 - (1.0)^2 + 1) = 2.25 + 0.5(2.25) = 3.375;$$

and

$$y(2) \approx w_4 = w_3 + 0.5(w_3 - (1.5)^2 + 1) = 3.375 + 0.5(2.125) = 4.4375. \quad \square$$

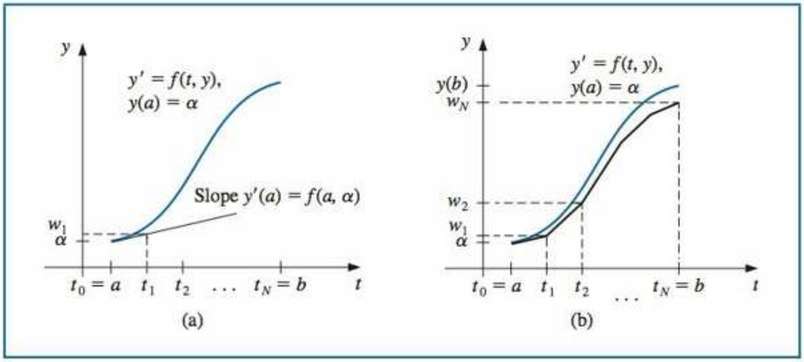
To interpret Euler's method geometrically, note that when w_i is a close approximation to $y(t_i)$, the assumption that the problem is well-posed implies that

$$f(t_i, w_i) \approx y'(t_i) = f(t_i, y(t_i)).$$

The program EULERM51 implements Euler's method.

The first step of Euler's method appears in Figure 5.2(a), and a series of steps appears in Figure 5.2(b).

Figure 5.2



Example 1 Euler's method was used in the Illustration with $h = 0.5$ to approximate the solution to the initial-value problem

$$y' = y - t^2 + 1, \quad 0 \le t \le 2, \quad y(0) = 0.5.$$

Use the program EULERM51 with $N = 10$ to determine approximations, and compare these with the exact values given by $y(t) = (t + 1)^2 - 0.5e^t$.

Solution With $N = 10$ we have $h = 0.2$, $t_i = 0.2i$, $w_0 = 0.5$, and

$$w_{i+1} = w_i + h\left(w_i - t_i^2 + 1\right) = w_i + 0.2\left[w_i - 0.04i^2 + 1\right] = 1.2w_i - 0.008i^2 + 0.2,$$

for $i = 0, 1, \ldots, 9$. So

$$w_1 = 1.2(0.5) - 0.008(0)^2 + 0.2 = 0.8; \quad w_2 = 1.2(0.8) - 0.008(1)^2 + 0.2 = 1.152;$$

and so on. Table 5.1 shows the comparison between the approximate values at t_i and the actual values. ■

Table 5.1

t_i	w_i	$y_i = y(t_i)$	$\lvert y_i - w_i \rvert$
0.0	0.5000000	0.5000000	0.0000000
0.2	0.8000000	0.8292986	0.0292986
0.4	1.1520000	1.2140877	0.0620877
0.6	1.5504000	1.6489406	0.0985406
0.8	1.9884800	2.1272295	0.1387495
1.0	2.4581760	2.6408591	0.1826831
1.2	2.9498112	3.1799415	0.2301303
1.4	3.4517734	3.7324000	0.2806266
1.6	3.9501281	4.2834838	0.3333557
1.8	4.4281538	4.8151763	0.3870225
2.0	4.8657845	5.3054720	0.4396874

Error Bounds for Euler's Method

Euler's method is derived from a Taylor polynomial whose error term involves the square of the step size h, so the local error at each step is proportional to h^2, so it is $O(h^2)$. However, the total error, or global error, accumulates these local errors, so it generally grows at a much faster rate.

Euler's Method Error Bound

Let $y(t)$ denote the unique solution to the initial-value problem

$$y' = f(t, y), \quad \text{for } a \le t \le b, \quad \text{with } y(a) = \alpha,$$

and $w_0, w_1, \ldots, w_N$ be the approximations generated by Euler's method for some positive integer N. Suppose that f is continuous for all t in $[a, b]$ and all y in $(-\infty, \infty)$, and constants L and M exist with

$$\left| \frac{\partial f}{\partial y}(t, y(t)) \right| \le L \quad \text{and} \quad |y''(t)| \le M.$$

Then, for each $i = 0, 1, 2, \ldots, N$,

$$|y(t_i) - w_i| \le \frac{hM}{2L}[e^{L(t_i - a)} - 1].$$

An important point to notice is that, although the local error of Euler's method, that is, the error at an individual step, is $O(h^2)$, the global error, which is the error over the entire interval, is only $O(h)$. The reduction of one power of h from local to global error is typical of initial-value techniques. Even though we have a reduction in order from local to global errors, the formula shows that the error tends to zero with h.

Example 2 The solution to the initial-value problem

$$y' = y - t^2 + 1, \quad 0 \le t \le 2, \quad y(0) = 0.5,$$

was approximated in Example 1 using Euler's method with $h = 0.2$. Find bounds for the approximation errors and compare these to the actual errors.

Solution Because $f(t, y) = y - t^2 + 1$, we have $\partial f(t, y)/\partial y = 1$ for all y, so $L = 1$. For this problem, the exact solution is $y(t) = (t+1)^2 - 0.5e^t$, so $y''(t) = 2 - 0.5e^t$ and

$$|y''(t)| \leq 0.5e^2 - 2, \quad \text{for all } t \in [0, 2].$$

Using the inequality in the error bound for Euler's method with $h = 0.2$, $L = 1$, and $M = 0.5e^2 - 2$ gives

$$|y_i - w_i| \leq 0.1(0.5e^2 - 2)(e^{t_i} - 1).$$

Hence

$$|y(0.2) - w_1| \leq 0.1(0.5e^2 - 2)(e^{0.2} - 1) = 0.03752;$$

$$|y(0.4) - w_2| \leq 0.1(0.5e^2 - 2)(e^{0.4} - 1) = 0.08334;$$

and so on. Table 5.2 lists the actual error found in Example 1, together with this error bound. Note that even though the true bound for the second derivative of the solution was used, the error bound is considerably larger than the actual error, especially for increasing values of t. ∎

Table 5.2

t_i	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0
Actual Error	0.02930	0.06209	0.09854	0.13875	0.18268	0.23013	0.28063	0.33336	0.38702	0.43969
Error Bound	0.03752	0.08334	0.13931	0.20767	0.29117	0.39315	0.51771	0.66985	0.85568	1.08264

Higher Order Taylor Methods

Euler's method was derived using Taylor's Theorem with $n = 1$, so the first attempt to find methods for improving the accuracy of difference methods is to extend this technique of derivation to larger values of n. Suppose the solution $y(t)$ to the initial-value problem

$$y' = f(t, y), \quad \text{for } a \leq t \leq b, \quad \text{with } y(a) = \alpha,$$

has $n + 1$ continuous derivatives. If we expand the solution $y(t)$ in terms of its nth Taylor polynomial about t_i, we obtain

$$y(t_{i+1}) = y(t_i) + hy'(t_i) + \frac{h^2}{2}y''(t_i) + \cdots + \frac{h^n}{n!}y^{(n)}(t_i) + \frac{h^{n+1}}{(n+1)!}y^{(n+1)}(\xi_i)$$

for some number ξ_i in (t_i, t_{i+1}). Successive differentiation of the solution $y(t)$ gives

$$y'(t) = f(t, y(t)), \quad y''(t) = f'(t, y(t)), \quad \text{and, generally,} \quad y^{(k)}(t) = f^{(k-1)}(t, y(t)).$$

Substituting these results into the Taylor expansion gives

$$y(t_{i+1}) = y(t_i) + hf(t_i, y(t_i)) + \frac{h^2}{2}f'(t_i, y(t_i)) + \cdots$$
$$+ \frac{h^n}{n!}f^{(n-1)}(t_i, y(t_i)) + \frac{h^{n+1}}{(n+1)!}f^{(n)}(\xi_i, y(\xi_i)).$$

The difference-equation method corresponding to this equation is obtained by deleting the remainder term involving ξ_i.

Taylor Method of Order n

$$w_0 = \alpha,$$
$$w_{i+1} = w_i + hT^{(n)}(t_i, w_i)$$

for each $i = 0, 1, \ldots, N-1$, where

$$T^{(n)}(t_i, w_i) = f(t_i, w_i) + \frac{h}{2}f'(t_i, w_i) + \cdots + \frac{h^{n-1}}{n!}f^{(n-1)}(t_i, w_i).$$

The local error is $\frac{1}{(n+1)!}y^{(n+1)}(\xi_i)h^{n+1}$ for some ξ_i in (t_i, t_{i+1}).

The formula for $T^{(n)}$ is easily expressed but difficult to use because it requires the derivatives of f with respect to t. Since f is described as a multivariable function of both t and y, the chain rule implies that the total derivative of f with respect to t, which we denoted $f'(t, y(t))$, is obtained by

$$f'(t, y(t)) = \frac{\partial f}{\partial t}(t, y(t)) \cdot \frac{dt}{dt} + \frac{\partial f}{\partial y}(t, y(t))\frac{dy(t)}{dt} = \frac{\partial f}{\partial t}(t, y(t)) + \frac{\partial f}{\partial y}(t, y(t))y'(t)$$

or, since $y'(t) = f(t, y(t))$, by

$$f'(t, y(t)) = \frac{\partial f}{\partial t}(t, y(t)) + f(t, y(t))\frac{\partial f}{\partial y}(t, y(t)).$$

Higher derivatives can be obtained in a similar manner, but they might become increasingly complicated. For example, $f''(t, y(t))$ involves the partial derivatives of all the terms on the right side of this equation with respect to both t and y.

Example 3 Apply Taylor's method of orders **(a)** 2 and **(b)** 4 with $N = 10$ to the initial-value problem

$$y' = y - t^2 + 1, \quad 0 \le t \le 2, \quad y(0) = 0.5.$$

Solution **(a)** For the method of order 2 we need the first derivative of $f(t, y(t)) = y(t) - t^2 + 1$ with respect to the variable t. Because $y' = y - t^2 + 1$ we have

$$f'(t, y(t)) = \frac{d}{dt}(y - t^2 + 1) = y' - 2t = y - t^2 + 1 - 2t,$$

so

$$T^{(2)}(t_i, w_i) = f(t_i, w_i) + \frac{h}{2}f'(t_i, w_i) = w_i - t_i^2 + 1 + \frac{h}{2}(w_i - t_i^2 + 1 - 2t_i)$$
$$= \left(1 + \frac{h}{2}\right)(w_i - t_i^2 + 1) - ht_i.$$

Because $N = 10$ we have $h = 0.2$, and $t_i = 0.2i$ for each $i = 1, 2, \ldots, 10$. Thus the second-order method becomes

$$w_0 = 0.5,$$
$$w_{i+1} = w_i + h\left[\left(1 + \frac{h}{2}\right)(w_i - t_i^2 + 1) - ht_i\right]$$
$$= w_i + 0.2\left[\left(1 + \frac{0.2}{2}\right)(w_i - 0.04i^2 + 1) - 0.04i\right]$$
$$= 1.22w_i - 0.0088i^2 - 0.008i + 0.22.$$

Table 5.3

t_i	Taylor Order 2 w_i	Error $\lvert y(t_i) - w_i \rvert$
0.0	0.500000	0
0.2	0.830000	0.000701
0.4	1.215800	0.001712
0.6	1.652076	0.003135
0.8	2.132333	0.005103
1.0	2.648646	0.007787
1.2	3.191348	0.011407
1.4	3.748645	0.016245
1.6	4.306146	0.022663
1.8	4.846299	0.031122
2.0	5.347684	0.042212

The first two steps give the approximations

$$y(0.2) \approx w_1 = 1.22(0.5) - 0.0088(0)^2 - 0.008(0) + 0.22 = 0.83;$$

$$y(0.4) \approx w_2 = 1.22(0.83) - 0.0088(0.2)^2 - 0.008(0.2) + 0.22 = 1.2158.$$

All the approximations and their errors are shown in Table 5.3.

(b) For Taylor's method of order 4 we need the first three derivatives of $f(t, y(t))$ with respect to t. Again using $y' = y - t^2 + 1$ we have

$$f'(t, y(t)) = y - t^2 + 1 - 2t,$$

$$f''(t, y(t)) = \frac{d}{dt}(y - t^2 + 1 - 2t) = y' - 2t - 2 = y - t^2 + 1 - 2t - 2$$
$$= y - t^2 - 2t - 1,$$

and

$$f'''(t, y(t)) = \frac{d}{dt}(y - t^2 - 2t - 1) = y' - 2t - 2 = y - t^2 - 2t - 1,$$

so

$$\begin{aligned} T^{(4)}(t_i, w_i) &= f(t_i, w_i) + \frac{h}{2} f'(t_i, w_i) + \frac{h^2}{6} f''(t_i, w_i) + \frac{h^3}{24} f'''(t_i, w_i) \\ &= w_i - t_i^2 + 1 + \frac{h}{2}\left(w_i - t_i^2 + 1 - 2t_i\right) + \frac{h^2}{6}\left(w_i - t_i^2 - 2t_i - 1\right) \\ &\quad + \frac{h^3}{24}\left(w_i - t_i^2 - 2t_i - 1\right) \\ &= \left(1 + \frac{h}{2} + \frac{h^2}{6} + \frac{h^3}{24}\right)(w_i - t_i^2) - \left(1 + \frac{h}{3} + \frac{h^2}{12}\right)(ht_i) + 1 + \frac{h}{2} - \frac{h^2}{6} - \frac{h^3}{24}. \end{aligned}$$

Hence Taylor's method of order 4 is

$$w_0 = 0.5,$$

$$w_{i+1} = w_i + h\left[\left(1 + \frac{h}{2} + \frac{h^2}{6} + \frac{h^3}{24}\right)(w_i - t_i^2) - \left(1 + \frac{h}{3} + \frac{h^2}{12}\right)ht_i + 1 + \frac{h}{2} - \frac{h^2}{6} - \frac{h^3}{24}\right],$$

for $i = 0, 1, \ldots, N - 1$.

Because $N = 10$ and $h = 0.2$ the method becomes

$$\begin{aligned} w_{i+1} = w_i + 0.2\Bigg[&\left(1 + \frac{0.2}{2} + \frac{0.04}{6} + \frac{0.008}{24}\right)(w_i - 0.04i^2) \\ &- \left(1 + \frac{0.2}{3} + \frac{0.04}{12}\right)(0.04i) + 1 + \frac{0.2}{2} - \frac{0.04}{6} - \frac{0.008}{24}\Bigg] \\ = 1.2214w_i &- 0.008856i^2 - 0.00856i + 0.2186, \end{aligned}$$

for each $i = 0, 1, \ldots, 9$. The first two steps give the approximations

$$y(0.2) \approx w_1 = 1.2214(0.5) - 0.008856(0)^2 - 0.00856(0) + 0.2186 = 0.8293;$$

$$y(0.4) \approx w_2 = 1.2214(0.8293) - 0.008856(0.2)^2 - 0.00856(0.2) + 0.2186 = 1.214091.$$

All the approximations and their errors are shown in Table 5.4. ■

Table 5.4

t_i	Taylor Order 4 w_i	Error $\lvert y(t_i) - w_i \rvert$
0.0	0.500000	0
0.2	0.829300	0.000001
0.4	1.214091	0.000003
0.6	1.648947	0.000006
0.8	2.127240	0.000010
1.0	2.640874	0.000015
1.2	3.179964	0.000023
1.4	3.732432	0.000032
1.6	4.283529	0.000045
1.8	4.815238	0.000062
2.0	5.305555	0.000083

Compare these results with those of Taylor's method of order 2 in Table 5.3 and you will see that the order 4 results are vastly superior.

Approximating Intermediate Results

Hermite interpolation requires both the value of the function and its derivative at each node. This makes it a natural interpolation method for approximating differential equations because these data are all available.

The results from Table 5.4 indicate the Taylor's method of order 4 results are quite accurate at the nodes 0.2, 0.4, etc. But suppose we need to determine an approximation to an intermediate point in the table, for example, at $t = 1.25$. If we use linear interpolation on the Taylor method of order four approximations at $t = 1.2$ and $t = 1.4$, we have

$$y(1.25) \approx \left(\frac{1.25 - 1.4}{1.2 - 1.4}\right)3.179964 + \left(\frac{1.25 - 1.2}{1.4 - 1.2}\right)3.732432 = 3.318081.$$

The true value is $y(1.25) = 3.317329$, so this approximation has an error of 0.000752, which is nearly 30 times the average of the approximation errors at 1.2 and 1.4.

We can significantly improve the approximation by using cubic Hermite interpolation. To determine this approximation for $y(1.25)$ requires approximations to $y'(1.2)$ and $y'(1.4)$ as well as approximations to $y(1.2)$ and $y(1.4)$. However, the approximations for $y(1.2)$ and $y(1.4)$ are in the table, and the derivative approximations are available from the differential equation because $y'(t) = f(t, y(t))$. In our example $y'(t) = y(t) - t^2 + 1$, so

$$y'(1.2) = y(1.2) - (1.2)^2 + 1 \approx 3.179964 - 1.44 + 1 = 2.739964$$

and

$$y'(1.4) = y(1.4) - (1.4)^2 + 1 \approx 3.732433 - 1.96 + 1 = 2.772432.$$

The divided-difference procedure in Section 3.4 gives the information in Table 5.5. The underlined entries come from the data, and the other entries use the divided-difference formulas.

Table 5.5

1.2	<u>3.179964</u>			
		<u>2.739964</u>		
1.2	<u>3.179964</u>		0.111880	
		2.762340		−0.307100
1.4	<u>3.732432</u>		0.050460	
		<u>2.772432</u>		
1.4	<u>3.732432</u>			

The cubic Hermite polynomial is

$$\begin{aligned} y(t) &\approx 3.179964 + (t - 1.2)2.739964 + (t - 1.2)^2 0.111880 \\ &\quad + (t - 1.2)^2(t - 1.4)(-0.307100), \end{aligned}$$

so

$$y(1.25) \approx 3.179964 + 0.136998 + 0.000280 + 0.000115 = 3.317357,$$

a result that is accurate to within 0.000028. This is about the average of the errors at 1.2 and at 1.4, and only 4% of the error obtained using linear interpolation. This improvement in accuracy certainly justifies the added computation required for the Hermite method.

Error estimates for the Taylor methods are similar to those for Euler's method. If sufficient differentiability conditions are met, an nth-order Taylor method will have local error $O(h^{n+1})$ and global error $O(h^n)$.

EXERCISE SET 5.2

1. Use Euler's method to approximate the solutions for each of the following initial-value problems.
   a. $y' = te^{3t} - 2y, \quad \text{for } 0 \le t \le 1, \quad \text{with } y(0) = 0 \text{ and } h = 0.5$
   b. $y' = 1 + (t - y)^2, \quad \text{for } 2 \le t \le 3, \quad \text{with } y(2) = 1 \text{ and } h = 0.5$
   c. $y' = 1 + \dfrac{y}{t}, \quad \text{for } 1 \le t \le 2, \quad \text{with } y(1) = 2 \text{ and } h = 0.25$
   d. $y' = \cos 2t + \sin 3t, \quad \text{for } 0 \le t \le 1, \quad \text{with } y(0) = 1 \text{ and } h = 0.25$
2. The actual solutions to the initial-value problems in Exercise 1 are given here. Compare the actual error at each step to the error bound.
   a. $y(t) = \dfrac{1}{5}te^{3t} - \dfrac{1}{25}e^{3t} + \dfrac{1}{25}e^{-2t}$
   b. $y(t) = t + (1 - t)^{-1}$
   c. $y(t) = t \ln t + 2t$
   d. $y(t) = \dfrac{1}{2}\sin 2t - \dfrac{1}{3}\cos 3t + \dfrac{4}{3}$
3. Use Euler's method to approximate the solutions for each of the following initial-value problems.
   a. $y' = \dfrac{y}{t} - \left(\dfrac{y}{t}\right)^2, \quad \text{for } 1 \le t \le 2, \quad \text{with } y(1) = 1 \text{ and } h = 0.1$
   b. $y' = 1 + \dfrac{y}{t} + \left(\dfrac{y}{t}\right)^2, \quad \text{for } 1 \le t \le 3, \quad \text{with } y(1) = 0 \text{ and } h = 0.2$
   c. $y' = -(y + 1)(y + 3), \quad \text{for } 0 \le t \le 2, \quad \text{with } y(0) = -2 \text{ and } h = 0.2$
   d. $y' = -5y + 5t^2 + 2t, \quad \text{for } 0 \le t \le 1, \quad \text{with } y(0) = 1/3 \text{ and } h = 0.1$
4. The actual solutions to the initial-value problems in Exercise 3 are given here. Compute the actual error in the approximations of Exercise 3.
   a. $y(t) = t(1 + \ln t)^{-1}$
   b. $y(t) = t \tan(\ln t)$
   c. $y(t) = -3 + 2(1 + e^{-2t})^{-1}$
   d. $y(t) = t^2 + \dfrac{1}{3}e^{-5t}$
5. Repeat Exercise 1 using Taylor's method of order 2.
6. Repeat Exercise 3 using Taylor's method of order 2.
7. Repeat Exercise 3 using Taylor's method of order 4.
8. Given the initial-value problem

$$y' = \frac{2}{t}y + t^2e^t, \quad 1 \le t \le 2, \quad y(1) = 0$$

   with the exact solution $y(t) = t^2(e^t - e)$:
   a. Use Euler's method with $h = 0.1$ to approximate the solution and compare it with the actual values of y.
   b. Use the answers generated in (a) and linear interpolation to approximate the following values of y and compare them to the actual values.
      i. $y(1.04)$ ii. $y(1.55)$ iii. $y(1.97)$
   c. Use Taylor's method of order 2 with $h = 0.1$ to approximate the solution and compare it with the actual values of y.
   d. Use the answers generated in (c) and linear interpolation to approximate y at the following values and compare them to the actual values of y.
      i. $y(1.04)$ ii. $y(1.55)$ iii. $y(1.97)$
   e. Use Taylor's method of order 4 with $h = 0.1$ to approximate the solution and compare it with the actual values of y.

f. Use the answers generated in (e) and piecewise cubic Hermite interpolation to approximate y at the following values and compare them to the actual values of y.

i. $y(1.04)$ **ii.** $y(1.55)$ **iii.** $y(1.97)$

9. Given the initial-value problem

$$y' = \frac{1}{t^2} - \frac{y}{t} - y^2, \quad 1 \le t \le 2, \quad y(1) = -1$$

with the exact solution $y(t) = -1/t$.

a. Use Euler's method with $h = 0.05$ to approximate the solution and compare it with the actual values of y.

b. Use the answers generated in (a) and linear interpolation to approximate the following values of y and compare them to the actual values.

i. $y(1.052)$ **ii.** $y(1.555)$ **iii.** $y(1.978)$

c. Use Taylor's method of order 2 with $h = 0.05$ to approximate the solution and compare it with the actual values of y.

d. Use the answers generated in (c) and linear interpolation to approximate the following values of y and compare them to the actual values.

i. $y(1.052)$ **ii.** $y(1.555)$ **iii.** $y(1.978)$

e. Use Taylor's method of order 4 with $h = 0.05$ to approximate the solution and compare it with the actual values of y.

f. Use the answers generated in (e) and piecewise cubic Hermite interpolation to approximate the following values of y and compare them to the actual values.

i. $y(1.052)$ **ii.** $y(1.555)$ **iii.** $y(1.978)$

10. In an electrical circuit with impressed voltage $\mathcal{E}$, having resistance R, inductance L, and capacitance C in parallel, the current i satisfies the differential equation

$$\frac{di}{dt} = C\frac{d^2\mathcal{E}}{dt^2} + \frac{1}{R}\frac{d\mathcal{E}}{dt} + \frac{1}{L}\mathcal{E}.$$

Suppose $i(0) = 0$, $C = 0.3$ farads, $R = 1.4$ ohms, $L = 1.7$ henries, and the voltage is given by

$$\mathcal{E}(t) = e^{-0.06\pi t}\sin(2t - \pi).$$

Use Euler's method to find the current i for the values $t = 0.1j$, $j = 0, 1, \ldots, 100$.

11. A projectile of mass $m = 0.11$ kg shot vertically upward with initial velocity $v(0) = 8$ m/s is slowed due to the force of gravity $F_g = mg$ and due to air resistance $F_r = -kv|v|$, where $g = -9.8$ m/s^2 and $k = 0.002$ kg/m. The differential equation for the velocity v is given by

$$mv' = mg - kv|v|.$$

a. Find the velocity after $0.1, 0.2, \ldots, 1.0$ s.

b. To the nearest tenth of a second, determine when the projectile reaches its maximum height and begins falling.

5.3 Runge-Kutta Methods

In the last section we saw how Taylor methods of arbitrary high order can be generated. However, the application of these high-order methods to a specific problem is complicated by the need to determine and evaluate high-order derivatives with respect to t on the right side of the differential equation. The widespread use of computer algebra systems has simplified this process, but it still remains cumbersome.