9. Suppose that

$$\begin{aligned} 2x_1 + x_2 + 3x_3 &= 1 \\ 4x_1 + 6x_2 + 8x_3 &= 5 \\ 6x_1 + \alpha x_2 + 10x_3 &= 5 \end{aligned}$$

with $|\alpha| < 10$. For which of the following values of α will there be no row interchange required when solving this system using scaled partial pivoting?

a. $\alpha = 6$ **b.** $\alpha = 9$ **c.** $\alpha = -3$

6.4 Linear Algebra and Matrix Inversion

Early in this chapter we illustrated the convenience of matrix notation for the study of linear systems of equations, but there is a wealth of additional material in linear algebra that finds application in the study of approximation techniques. In this section we introduce some basic notation and results that are needed for both theory and application. All the topics discussed here should be familiar to anyone who has studied matrix theory at the undergraduate level. This section could then be omitted, but it is advisable to read the section to see the results from linear algebra that will be frequently called upon for service, and to be aware of the notation being used.

Two matrices A and B are **equal** if both are of the same size, say, $n \times m$, and if $a_{ij} = b_{ij}$ for each $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$.

This definition means, for example, that

$$\begin{bmatrix} 2 & -1 & 7 \\ 3 & 1 & 0 \end{bmatrix} \neq \begin{bmatrix} 2 & 3 \\ -1 & 1 \\ 7 & 0 \end{bmatrix}$$

because they differ in dimension.

Matrix Arithmetic

If A and B are $n \times m$ matrices and λ is a real number, then

- the **sum** of A and B, denoted $A + B$, is the $n \times m$ matrix whose entries are $a_{ij} + b_{ij}$,
- the **scalar product** of λ and A, denoted λA, is the $n \times m$ matrix whose entries are λa_{ij}.

Example 1 Determine $A + B$ and λA when

$$A = \begin{bmatrix} 2 & -1 & 7 \\ 3 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & 2 & -8 \\ 0 & 1 & 6 \end{bmatrix}, \quad \text{and } \lambda = -2.$$

Solution We have

$$A + B = \begin{bmatrix} 2+4 & -1+2 & 7-8 \\ 3+0 & 1+1 & 0+6 \end{bmatrix} = \begin{bmatrix} 6 & 1 & -1 \\ 3 & 2 & 6 \end{bmatrix},$$

and

$$\lambda A = \begin{bmatrix} -2(2) & -2(-1) & -2(7) \\ -2(3) & -2(1) & -2(0) \end{bmatrix} = \begin{bmatrix} -4 & 2 & -14 \\ -6 & -2 & 0 \end{bmatrix}.$$

■

Matrix-Matrix Products

If A is an $n \times m$ matrix and B is an $m \times p$ matrix,

- the **matrix product** of A and B, denoted AB, is an $n \times p$ matrix C whose entries c_{ij} are given by

$$c_{ij} = \sum_{k=1}^{m} a_{ik}b_{kj} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{im}b_{mj},$$

for each $i = 1, 2, \ldots n$ and $j = 1, 2, \ldots, p$.

The computation of c_{ij} can be viewed as the multiplication of the entries of the ith row of A with corresponding entries in the jth column of B, followed by a summation; that is,

$$[a_{i1}, a_{i2}, \ldots, a_{im}] \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{mj} \end{bmatrix} = [c_{ij}],$$

where

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{im}b_{mj} = \sum_{k=1}^{m} a_{ik}b_{kj}.$$

This explains why the number of columns of A must equal the number of rows of B for the product AB to be defined.

The following example illustrates a common matrix multiplying operation in the case when the matrix on the right of the multiplication has only one column, that is, it is a column vector.

Example 2 Determine the product $A\mathbf{b}$ if $A = \begin{bmatrix} 3 & 2 \\ -1 & 1 \\ 6 & 4 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$.

Solution Because A has dimension 3×2 and $\mathbf{b}$ has dimension 2×1, the product is defined and is 3×1, that is, a vector with three rows. These are

$$3(3) + 2(-1) = 7, \quad (-1)(3) + 1(-1) = -4, \quad \text{and} \quad 6(3) + 4(-1) = 14.$$

So

$$A\mathbf{b} = \begin{bmatrix} 3 & 2 \\ -1 & 1 \\ 6 & 4 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 7 \\ -4 \\ 14 \end{bmatrix}$$ ■

In the next example we consider product operations in various situations.

Example 3 Determine all possible products of the matrices

$$A = \begin{bmatrix} 3 & 2 \\ -1 & 1 \\ 1 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 1 & -1 \\ 3 & 1 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & 1 & 0 & 1 \\ -1 & 3 & 2 & 1 \\ 1 & 1 & 2 & 0 \end{bmatrix}, \quad \text{and}$$

$$D = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}.$$

Solution The sizes of the matrices are

$$A\colon 3 \times 2, \quad B\colon 2 \times 3, \quad C\colon 3 \times 4, \quad \text{and} \quad D\colon 2 \times 2.$$

The products that can be defined, and their dimensions, are:

$$AB\colon 3 \times 3, \quad BA\colon 2 \times 2, \quad AD\colon 3 \times 2, \quad BC\colon 2 \times 4, \quad DB\colon 2 \times 3, \quad \text{and} \quad DD\colon 2 \times 2.$$

These products are

$$AB = \begin{bmatrix} 12 & 5 & 1 \\ 1 & 0 & 3 \\ 14 & 5 & 7 \end{bmatrix}, \qquad BA = \begin{bmatrix} 4 & 1 \\ 10 & 15 \end{bmatrix}, \qquad AD = \begin{bmatrix} 7 & -5 \\ 1 & 0 \\ 9 & -5 \end{bmatrix},$$

$$BC = \begin{bmatrix} 2 & 4 & 0 & 3 \\ 7 & 8 & 6 & 4 \end{bmatrix}, \quad DB = \begin{bmatrix} -1 & 0 & -3 \\ 1 & 1 & -4 \end{bmatrix}, \quad \text{and} \quad DD = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

■

Square Matrices

We have some special names and notation for matrices with the same number of rows and columns.

The term *diagonal* applied to a matrix refers to the entries in the diagonal that run from the top left entry to the bottom right entry.

- A **square** matrix has the same number of rows as columns.
- A **diagonal** matrix is a square matrix whose only nonzero elements are along the main diagonal. So if $D = [d_{ij}]$ is a diagonal matrix, then $d_{ij} = 0$ whenever $i \neq j$.
- The **identity** matrix of order n, $I_n = [\delta_{ij}]$, is a diagonal matrix with 1s along the diagonal. That is,

$$I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix}.$$

When the size of I_n is clear, this matrix is generally written simply as I. For example, the identity matrix of order three is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

If A is any $n \times n$ matrix and $I = I_n$, then $AI = IA = A$.

Illustration Consider the identity matrix of order three,

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

If A is any 3×3 matrix, then

$$AI_3 = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = A. \qquad \square$$

A triangular matrix is one that has all its nonzero entries either on and above (upper) or on and below (lower) the main diagonal. A matrix is diagonal precisely when it is both upper- and lower-triangular.

An $n \times n$ **upper-triangular** matrix $U = [u_{ij}]$ has all its nonzero entries on or above the main diagonal, that is, for each $j = 1, 2, \ldots, n$, the entries

$$u_{ij} = 0, \quad \text{for each } i = j+1, j+2, \ldots, n.$$

In a similar manner, a **lower-triangular** matrix $L = [l_{ij}]$ has all its nonzero entries on or below the main diagonal, that is, for each $j = 1, 2, \ldots, n$, the entries

$$l_{ij} = 0, \quad \text{for each } i = 1, 2, \ldots, j-1.$$

(A diagonal matrix is both upper and lower triangular.)

In Example 3 we found that, in general, $AB \neq BA$, even when both products are defined. However, the other arithmetic properties associated with multiplication do hold. For example, when A, B, and C are matrices of the appropriate size and λ is a scalar, we have

- $A(BC) = (AB)C$,
- $A(B + C) = AB + AC$, and
- $\lambda(AB) = (\lambda A)B = A(\lambda B)$.

Inverse Matrices

The word *singular* means something that deviates from the ordinary. Hence a singular matrix does *not* have an inverse.

Certain $n \times n$ matrices have the property that another $n \times n$ matrix, which we will denote A^{-1}, exists with $AA^{-1} = A^{-1}A = I$. In this case A is said to be **nonsingular**, or *invertible*, and the matrix A^{-1} is called the **inverse** of A. A matrix without an inverse is called **singular**, or *noninvertible*.

Example 4 Let

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 0 \\ -1 & 1 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -\frac{2}{9} & \frac{5}{9} & -\frac{1}{9} \\ \frac{4}{9} & -\frac{1}{9} & \frac{2}{9} \\ -\frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}.$$

Show that $B = A^{-1}$, and that the solution to the linear system described by

$$\begin{aligned} x_1 + 2x_2 - x_3 &= 2, \\ 2x_1 + x_2 \phantom{{}+2x_3} &= 3, \\ -x_1 + x_2 + 2x_3 &= 4. \end{aligned}$$

is given by the entries in $B\mathbf{b}$, where $\mathbf{b}$ is the column vector with entries 2, 3, and 4.

Solution First note that

$$AB = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 0 \\ -1 & 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} -\frac{2}{9} & \frac{5}{9} & -\frac{1}{9} \\ \frac{4}{9} & -\frac{1}{9} & \frac{2}{9} \\ -\frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3.$$

In a similar manner, $BA = I_3$, so A and B are both nonsingular with $B = A^{-1}$ and $A = B^{-1}$.

Now convert the given linear system to the matrix equation

$$\begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 0 \\ -1 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix},$$

and multiply both sides by B, the inverse of A. Then we have

$$B(A\mathbf{x}) = B\mathbf{b},$$

with

$$B(A\mathbf{x}) = (BA)\mathbf{x} = \left(\begin{bmatrix} -\frac{2}{9} & \frac{5}{9} & -\frac{1}{9} \\ \frac{4}{9} & -\frac{1}{9} & \frac{2}{9} \\ -\frac{3}{9} & \frac{3}{9} & \frac{3}{9} \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 0 \\ -1 & 1 & 2 \end{bmatrix} \right) \mathbf{x} = \mathbf{x}$$

and

$$B\mathbf{b} = \begin{bmatrix} -\frac{2}{9} & \frac{5}{9} & -\frac{1}{9} \\ \frac{4}{9} & -\frac{1}{9} & \frac{2}{9} \\ -\frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} \frac{7}{9} \\ \frac{13}{9} \\ \frac{5}{3} \end{bmatrix}.$$

This implies that $\mathbf{x} = B\mathbf{b}$ and gives the solution $x_1 = 7/9$, $x_2 = 13/9$, and $x_3 = 5/3$. ■

The reason for introducing this matrix operation at this time is that the linear system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2, \\ \vdots \qquad\qquad & \quad \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n, \end{aligned}$$

can be viewed as the matrix equation $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}.$$

If A is a nonsingular matrix, then the solution $\mathbf{x}$ to the linear system $A\mathbf{x} = \mathbf{b}$ is given by

$$\mathbf{x} = A^{-1}(A\mathbf{x}) = A^{-1}\mathbf{b}.$$

In general, however, it is more difficult to determine A^{-1} than it is to solve the system $A\mathbf{x} = \mathbf{b}$ because the number of operations involved in determining A^{-1} is larger. Even so, it is useful from a conceptual standpoint to describe a method for determining the inverse of a matrix.

Illustration To determine the inverse of the matrix

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 0 \\ -1 & 1 & 2 \end{bmatrix},$$

let us first consider the product AB, where B is an arbitrary 3×3 matrix.

$$AB = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 0 \\ -1 & 1 & 2 \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

$$= \begin{bmatrix} b_{11}+2b_{21}-b_{31} & b_{12}+2b_{22}-b_{32} & b_{13}+2b_{23}-b_{33} \\ 2b_{11}+b_{21} & 2b_{12}+b_{22} & 2b_{13}+b_{23} \\ -b_{11}+b_{21}+2b_{31} & -b_{12}+b_{22}+2b_{32} & -b_{13}+b_{23}+2b_{33} \end{bmatrix}.$$

If $B = A^{-1}$, then $AB = I$, so we must have

$$\begin{aligned} b_{11}+2b_{21}-\ b_{31}&=1, \\ 2b_{11}+\ b_{21} \qquad &=0, \\ -b_{11}+\ b_{21}+2b_{31}&=0, \end{aligned} \qquad \begin{aligned} b_{12}+2b_{22}-\ b_{32}&=0, \\ 2b_{12}+\ b_{22} \qquad &=1, \\ -b_{12}+\ b_{22}+2b_{32}&=0, \end{aligned}$$

$$\begin{aligned} b_{13}+2b_{23}-\ b_{33}&=0 \\ 2b_{13}+\ b_{23} \qquad &=0 \\ -b_{13}+\ b_{23}+2b_{33}&=1 \end{aligned}$$

Notice that the coefficients in each of the systems of equations are the same; the only change in the systems occurs on the right side of the equations. As a consequence, the computations can be performed on the larger augmented matrix, which is formed by combining the matrices for each of the systems

$$\left[\begin{array}{rrr:rrr} 1 & 2 & -1 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ -1 & 1 & 2 & 0 & 0 & 1 \end{array}\right].$$

First, performing $(E_2 - 2E_1) \to (E_1)$ and $(E_3 + E_1) \to (E_3)$ gives

$$\left[\begin{array}{rrr:rrr} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & -3 & 2 & -2 & 1 & 0 \\ 0 & 3 & 1 & 1 & 0 & 1 \end{array}\right].$$

Next, performing $(E_3 + E_2) \to (E_3)$ produces

$$\left[\begin{array}{rrr:rrr} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & -3 & 2 & -2 & 1 & 0 \\ 0 & 0 & 3 & -1 & 1 & 1 \end{array}\right].$$

Backward substitution is performed on each of the three augmented matrices,

$$\left[\begin{array}{rrr:r} 1 & 2 & -1 & 1 \\ 0 & -3 & 2 & -2 \\ 0 & 0 & 3 & -1 \end{array}\right], \quad \left[\begin{array}{rrr:r} 1 & 2 & -1 & 0 \\ 0 & -3 & 2 & 1 \\ 0 & 0 & 3 & 1 \end{array}\right], \quad \left[\begin{array}{rrr:r} 1 & 2 & -1 & 0 \\ 0 & -3 & 2 & 0 \\ 0 & 0 & 3 & 1 \end{array}\right],$$

to eventually give

$$\begin{aligned} b_{11} &= -\tfrac{2}{9}, & b_{12} &= \tfrac{5}{9}, & & & b_{13} &= -\tfrac{1}{9}, \\ b_{21} &= \tfrac{4}{9}, & b_{22} &= -\tfrac{1}{9}, & &\text{and} & b_{23} &= \tfrac{2}{9}, \\ b_{31} &= -\tfrac{1}{3}, & b_{32} &= \tfrac{1}{3}, & & & b_{33} &= \tfrac{1}{3}. \end{aligned}$$

These are the entries of A^{-1}:

$$A^{-1} = \begin{bmatrix} -\frac{2}{9} & \frac{5}{9} & -\frac{1}{9} \\ \frac{4}{9} & -\frac{1}{9} & \frac{2}{9} \\ -\frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix} = \frac{1}{9} \begin{bmatrix} -2 & 5 & -1 \\ 4 & -1 & 2 \\ -3 & 3 & 3 \end{bmatrix}. \tag{6.3}$$

□

Transpose of a Matrix

- The **transpose** of an $n \times m$ matrix $A = [a_{ij}]$ is the $m \times n$ matrix $A^t = [a_{ji}]$.
- A square matrix A is **symmetric** if $A = A^t$.

Illustration The matrices

$$A = \begin{bmatrix} 7 & 2 & 0 \\ 3 & 5 & -1 \\ 0 & 5 & -6 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 4 & 7 \\ 3 & -5 & -1 \end{bmatrix}, \quad C = \begin{bmatrix} 6 & 4 & -3 \\ 4 & -2 & 0 \\ -3 & 0 & 1 \end{bmatrix}$$

have transposes

$$A^t = \begin{bmatrix} 7 & 3 & 0 \\ 2 & 5 & 5 \\ 0 & -1 & -6 \end{bmatrix}, \quad B^t = \begin{bmatrix} 2 & 3 \\ 4 & -5 \\ 7 & -1 \end{bmatrix}, \quad C^t = \begin{bmatrix} 6 & 4 & -3 \\ 4 & -2 & 0 \\ -3 & 0 & 1 \end{bmatrix}.$$

The matrix C is symmetric because $C^t = C$. The matrices A and B are not symmetric. □

The transpose notation is convenient for expressing column vectors in a more compact manner. Because the transpose of a column vector is a row vector,

the column vector $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ is generally written in text as $\mathbf{x} = (x_1, x_2, \ldots, x_n)^t$.

The following operations involving the transpose of a matrix hold whenever the operation is possible.

Transpose Facts

(i) $(A^t)^t = A$. **(ii)** $(A + B)^t = A^t + B^t$.

(iii) $(AB)^t = B^t A^t$. **(iv)** If A^{-1} exists, $(A^{-1})^t = (A^t)^{-1}$.

Matrix Determinants

The determinant of a square matrix is a number that can be useful in determining the existence and uniqueness of solutions to linear systems. We will denote the determinant of a matrix A by $\det A$, but it is also common to use the notation $|A|$.

Determinant of a Matrix

(i) If $A = [a]$ is a 1×1 matrix, then $\det A = a$.

(ii) If A is an $n \times n$ matrix, the **minor** M_{ij} is the determinant of the $(n-1) \times (n-1)$ submatrix of A obtained by deleting the ith row and jth column of the matrix A.

Then the determinant of A is given either by

$$\det A = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} M_{ij} \quad \text{for any } i = 1, 2, \ldots, n,$$

or by

$$\det A = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} M_{ij} \quad \text{for any } j = 1, 2, \ldots, n.$$

The notion of a determinant appeared independently in 1683 in Japan and Europe, although neither Takakazu Seki Kowa (1642–1708) nor Gottfried Leibniz (1646–1716) appear to have used the term determinant.

To calculate the determinant of a general $n \times n$ matrix by expanding by minors requires $O(n!)$ multiplications/divisions and additions/subtractions. Even for relatively small values of n, the number of calculations becomes unwieldy. Fortunately, the precise value of the determinant is seldom needed, and there are efficient ways to approximate its value.

Although it appears that there are $2n$ different definitions of $\det A$, depending on which row or column is chosen, all definitions give the same numerical result. The flexibility in the definition is used in the following example. It is most convenient to compute $\det A$ across the row or down the column with the most zeros.

Example 5 Find the determinant of the matrix

$$A = \begin{bmatrix} 2 & -1 & 3 & 0 \\ 4 & -2 & 7 & 0 \\ -3 & -4 & 1 & 5 \\ 6 & -6 & 8 & 0 \end{bmatrix}$$

using the row or column with the most zero entries.

Solution To compute $\det A$, it is easiest to use the fourth column because three of its entries are 0.

$$\det A = a_{14}(-1)^5 M_{14} + a_{24}(-1)^6 M_{24} + a_{34}(-1)^7 M_{34} + a_{44}(-1)^8 M_{44} = -5M_{34}.$$

Eliminating the third row and the fourth column of A and expanding the resulting 3×3 matrix by its first row gives

$$\begin{aligned} \det A &= -5 \det \begin{bmatrix} 2 & -1 & 3 \\ 4 & -2 & 7 \\ 6 & -6 & 8 \end{bmatrix} \\ &= -5 \left\{ 2 \det \begin{bmatrix} -2 & 7 \\ -6 & 8 \end{bmatrix} - (-1) \det \begin{bmatrix} 4 & 7 \\ 6 & 8 \end{bmatrix} + 3 \det \begin{bmatrix} 4 & -2 \\ 6 & -6 \end{bmatrix} \right\} \\ &= -5\,[2(-16+42) + (32-42) + 3(-24+12)] = -30. \end{aligned}$$

▪

The following properties of determinants are useful in relating linear systems and Gaussian elimination to determinants.

Determinant Facts

Suppose A is an $n \times n$ matrix:

(i) If any row or column of A has only zero entries, then $\det A = 0$.

(ii) If $\tilde{A}$ is obtained from A by the operation $(E_i) \leftrightarrow (E_k)$, with $i \neq k$, then $\det \tilde{A} = -\det\ A$.

(iii) If A has two rows or two columns the same, then $\det A = 0$.

(iv) If $\tilde{A}$ is obtained from A by the operation $(\lambda E_i) \to (E_i)$, then $\det \tilde{A} = \lambda \det A$.

(v) If $\tilde{A}$ is obtained from A by the operation $(E_i + \lambda E_k) \to (E_i)$ with $i \neq k$, then $\det \tilde{A} = \det A$.

(vi) If B is also an $n \times n$ matrix, then $\det AB = \det A \cdot \det B$.

(vii) $\det A^t = \det A$.

(viii) If A^{-1} exists, then $\det A^{-1} = \dfrac{1}{\det A}$.

(ix) If A is an upper triangular, lower triangular, or diagonal matrix, then

$$\det A = a_{11} \cdot a_{22} \cdots a_{nn}.$$

Example 6 Compute the determinant of the matrix

$$A = \begin{bmatrix} 2 & 1 & -1 & 1 \\ 1 & 1 & 0 & 3 \\ -1 & 2 & 3 & -1 \\ 3 & -1 & -1 & 2 \end{bmatrix}$$

using Determinant Facts (ii), (iv), (v), and (ix), and doing the computations in MATLAB.

Solution Matrix A is defined in MATLAB by

```
A=[2 1 -1 1; 1 1 0 3; -1 2 3 -1; 3 -1 -1 2]
```

We will use the operations in Table 6.2 to first place the matrix in upper-triangular form. Then we can use the final determinant fact to contain the result from the entries on the diagonal. We have used some of these steps to illustrate the commands in MATLAB. For example, the first operation would not normally be performed when placing the matrix in upper-triangular form.

Table 6.2

Operation	MATLAB Command	Effect
$\frac{1}{2}E_1 \to E_1$	`A1(1,:) = A(1,:)/2`	$\det A1 = \frac{1}{2}\det A$
$E_2 - E_1 \to E_2$	`A2(2,:)=A1(2,:)-A1(1,:)`	$\det A2 = \det A1 = \frac{1}{2}\det A$
$E_3 + E_1 \to E_3$	`A3(3,:)=A2(3,:)+A2(1,:)`	$\det A3 = \det A2 = \frac{1}{2}\det A$
$E_4 - 3E_1 \to E_4$	`A4(4,:)=A3(4,:)-3*A3(1,:)`	$\det A4 = \det A3 = \frac{1}{2}\det A$
$2E_2 \to E_2$	`A5(2,:)=2*A4(2,:)`	$\det A5 = 2\det A4 = \det A$
$E_3 - \frac{5}{2}E_2 \to E_3$	`A6(3,:)=A5(3,:)-2.5*A5(2,:)`	$\det A6 = \det A5 = \det A$
$E_4 + \frac{5}{2}E_2 \to E_4$	`A7(4,:)=A6(4,:)+2.5*A6(2,:)`	$\det A7 = \det A6 = \det A$
$E_3 \leftrightarrow E_4$	`B=A7(3,:),A8(3,:)=A7(4,:),A8(4,:)=B`	$\det A8 = -\det A7 = -\det A$

After these operations are performed, the matrix will have the form

$$A8 = \begin{bmatrix} 1 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 1 & 5 \\ 0 & 0 & 3 & 13 \\ 0 & 0 & 0 & -13 \end{bmatrix}$$

By (ix), $\det A8 = 1 \cdot 1 \cdot 3(-13) = -39$, so $\det A = -\det A8 = 39$. ■

The key result relating nonsingularity, Gaussian elimination, linear systems, and determinants is that the following statements are equivalent.

Equivalent Statements about an $n \times n$ Matrix A

(i) The equation $A\mathbf{x} = \mathbf{0}$ has the unique solution $\mathbf{x} = \mathbf{0}$.

(ii) The system $A\mathbf{x} = \mathbf{b}$ has a unique solution for any n-dimensional column vector $\mathbf{b}$.

(iii) The matrix A is nonsingular; that is, A^{-1} exists.

(iv) $\det A \neq 0$.

(v) Gaussian elimination with row interchanges can be performed on the system $A\mathbf{x} = \mathbf{b}$ for any n-dimensional column vector $\mathbf{b}$ to find the unique solution $\mathbf{x}$.

MATLAB has numerous commands that can directly perform operations on matrices. For example, for matrices A and B and scalar a, when the operations are possible, the following MATLAB commands can be used.

- To add A and B: `A+B`
- To multiply A by a: `a*A`
- To obtain the inverse of A: `inv(A)`
- To multiply A and B: `A*B`
- To obtain the transpose of A: `A'`
- To find the determinant of A: `det(A)`

EXERCISE SET 6.4

1. Compute the following matrix products.

a. $\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 1 & -1 & 1 \end{bmatrix}$

b. $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -2 & -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{bmatrix}$

c. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$

d. $\begin{bmatrix} 2 & -1 & 4 \\ 0 & -1 & 2 \\ 0 & 0 & 3 \end{bmatrix} \cdot \begin{bmatrix} 3 & -3 & 4 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$

2. For the following matrices:

i. Find the transpose of the matrix.

ii. Determine which matrices are nonsingular and compute their inverses.

a. $\begin{bmatrix} 4 & 2 & 6 \\ 3 & 0 & 7 \\ -2 & -1 & -3 \end{bmatrix}$

b. $\begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & -1 \\ 3 & 1 & 1 \end{bmatrix}$

c. $\begin{bmatrix} 4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

d. $\begin{bmatrix} 1 & 1 & -1 & 1 \\ 1 & 2 & -4 & -2 \\ 2 & 1 & 1 & 5 \\ -1 & 0 & -2 & -4 \end{bmatrix}$

e. $\begin{bmatrix} 4 & 0 & 0 & 0 \\ 6 & 7 & 0 & 0 \\ 9 & 11 & 1 & 0 \\ 5 & 4 & 1 & 1 \end{bmatrix}$

f. $\begin{bmatrix} 2 & 0 & 1 & 2 \\ 1 & 1 & 0 & 2 \\ 2 & -1 & 3 & 1 \\ 3 & -1 & 4 & 3 \end{bmatrix}$

3. Compute the determinants of the matrices in Exercise 2 and the determinants of the inverse matrices of those that are nonsingular.

4. Consider the four 3×3 linear systems having the same coefficient matrix:

$$\begin{aligned} 2x_1 - 3x_2 + x_3 &= 2, \\ x_1 + x_2 - x_3 &= -1, \\ -x_1 + x_2 - 3x_3 &= 0, \end{aligned} \qquad \begin{aligned} 2x_1 - 3x_2 + x_3 &= 6, \\ x_1 + x_2 - x_3 &= 4, \\ -x_1 + x_2 - 3x_3 &= 5, \end{aligned}$$

$$\begin{aligned} 2x_1 - 3x_2 + x_3 &= 0, \\ x_1 + x_2 - x_3 &= 1, \\ -x_1 + x_2 - 3x_3 &= -3, \end{aligned} \qquad \begin{aligned} 2x_1 - 3x_2 + x_3 &= -1, \\ x_1 + x_2 - x_3 &= 0, \\ -x_1 + x_2 - 3x_3 &= 0. \end{aligned}$$

a. Solve the linear systems by applying Gaussian elimination to the augmented matrix

$$\left[\begin{array}{rrr:rrrr} 2 & -3 & 1 & 2 & 6 & 0 & -1 \\ 1 & 1 & -1 & -1 & 4 & 1 & 0 \\ -1 & 1 & -3 & 0 & 5 & -3 & 0 \end{array}\right].$$

b. Solve the linear systems by finding and multiplying by the inverse of

$$A = \begin{bmatrix} 2 & -3 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & -3 \end{bmatrix}.$$

c. Which method requires more operations?

5. Show that the following statements are true or provide counterexamples to show they are not.

a. The product of two symmetric matrices is symmetric.

b. The inverse of a nonsingular symmetric matrix is a nonsingular symmetric matrix.

c. If A and B are $n \times n$ matrices, then $(AB)^t = A^t B^t$.

6. **a.** Show that the product of two $n \times n$ lower triangular matrices is lower triangular.

b. Show that the product of two $n \times n$ upper triangular matrices is upper triangular.

c. Show that the inverse of a nonsingular $n \times n$ lower triangular matrix is lower triangular.

7. The solution by **Cramer's rule** to the linear system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1, \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2, \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned}$$

has

$$x_1 = \frac{1}{D} \det \begin{bmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{bmatrix} \equiv \frac{D_1}{D},$$

$$x_2 = \frac{1}{D} \det \begin{bmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{bmatrix} \equiv \frac{D_2}{D},$$

and

$$x_3 = \frac{1}{D} \det \begin{bmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{bmatrix} \equiv \frac{D_3}{D},$$

where

$$D = \det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}.$$

a. Use Cramer's rule to find the solution to the linear system

$$\begin{aligned} 2x_1 + \ 3x_2 - \ \ x_3 &= 4, \\ x_1 - \ 2x_2 + \ \ x_3 &= 6, \\ x_1 - 12x_2 + 5x_3 &= 10. \end{aligned}$$

b. Show that the linear system

$$\begin{aligned} 2x_1 + \ 3x_2 - \ \ x_3 &= 4, \\ x_1 - \ 2x_2 + \ \ x_3 &= 6, \\ -x_1 - 12x_2 + 5x_3 &= 9 \end{aligned}$$

does not have a solution. Compute D_1, D_2, and D_3.

c. Show that the linear system

$$\begin{aligned} 2x_1 + \ 3x_2 - \ \ x_3 &= 4, \\ x_1 - \ 2x_2 + \ \ x_3 &= 6, \\ -x_1 - 12x_2 + 5x_3 &= 10 \end{aligned}$$

has an infinite number of solutions. Compute D_1, D_2, and D_3.

d. Suppose that a 3×3 linear system with $D = 0$ has solutions. Explain why we must also have $D_1 = D_2 = D_3 = 0$.

8. In a paper entitled "Population Waves," Bernadelli [Ber] hypothesizes a type of simplified beetle, which has a natural life span of 3 years. The female of this species has a survival rate of $\frac{1}{2}$ in the first year of life, has a survival rate of $\frac{1}{3}$ from the second to third years, and gives birth to an average of six new females before expiring at the end of the third year. A matrix can be used to show the contribution an individual female beetle makes, in a probabilistic sense, to the female population of the species by letting a_{ij} in the matrix $A = [a_{ij}]$ denote the contribution that a single female beetle of age j will make to the next year's female population of age i; that is,

$$A = \begin{bmatrix} 0 & 0 & 6 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \end{bmatrix}.$$

a. The contribution that a female beetle makes to the population 2 years hence is determined from the entries of A^2, of 3 years hence from A^3, and so on. Construct A^2 and A^3, and try to make a general statement about the contribution of a female beetle to the population in n years' time for any positive integral value of n.

b. Use your conclusions from part (a) to describe what will occur in future years to a population of these beetles that initially consists of 6000 female beetles in each of the three age groups.

c. Construct A^{-1} and describe its significance regarding the population of this species.

9. The study of food chains is an important topic in the determination of the spread and accumulation of environmental pollutants in living matter. Suppose that a food chain has three links. The first link consists of vegetation of types $v_1, v_2, \ldots, v_n$, which provide all the food requirements for herbivores of species $h_1, h_2, \ldots, h_m$ in the second link. The third link consists of carnivorous animals $c_1, c_2, \ldots, c_k$, which depend entirely on the herbivores in the second link for their food supply. The coordinate a_{ij} of the matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix}$$

represents the total number of plants of type v_i eaten by the herbivores in the species h_j, whereas b_{ij} in

$$B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1k} \\ b_{21} & b_{22} & \cdots & b_{2k} \\ \vdots & \vdots & & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mk} \end{bmatrix}$$

describes the number of herbivores in species h_i that are devoured by the animals of type c_j.

a. Show that the number of plants of type v_i that eventually end up in the animals of species c_j is given by the entry in the ith row and jth column of the matrix AB.

b. What physical significance is associated with the matrices A^{-1}, B^{-1}, and $(AB)^{-1} = B^{-1}A^{-1}$?

10. In Section 3.6 we found that the parametric form $(x(t), y(t))$ of the cubic Hermite polynomials through $(x(0), y(0)) = (x_0, y_0)$ and $(x(1), y(1)) = (x_1, y_1)$ with guidepoints $(x_0 + \alpha_0, y_0 + \beta_0)$ and $(x_1 - \alpha_1, y_1 - \beta_1)$, respectively, is given by

$$x(t) = [2(x_0 - x_1) + (\alpha_0 + \alpha_1)]t^3 + [3(x_1 - x_0) - \alpha_1 - 2\alpha_0]t^2 + \alpha_0 t + x_0$$

and

$$y(t) = [2(y_0 - y_1) + (\beta_0 + \beta_1)]t^3 + [3(y_1 - y_0) - \beta_1 - 2\beta_0]t^2 + \beta_0 t + y_0.$$

The Bézier cubic polynomials have the form

$$\hat{x}(t) = [2(x_0 - x_1) + 3(\alpha_0 + \alpha_1)]t^3 + [3(x_1 - x_0) - 3(\alpha_1 + 2\alpha_0)]t^2 + 3\alpha_0 t + x_0$$

and

$$\hat{y}(t) = [2(y_0 - y_1) + 3(\beta_0 + \beta_1)]t^3 + [3(y_1 - y_0) - 3(\beta_1 + 2\beta_0)]t^2 + 3\beta_0 t + y_0.$$

a. Show that the matrix

$$A = \begin{bmatrix} 7 & 4 & 4 & 0 \\ -6 & -3 & -6 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

maps the Hermite polynomial coefficients onto the Bézier polynomial coefficients.

b. Determine a matrix B that maps the Bézier polynomial coefficients onto the Hermite polynomial coefficients.

11. Consider the 2×2 linear system $(A + iB)(\mathbf{x} + i\mathbf{y}) = \mathbf{c} + i\mathbf{d}$ with complex entries in component form:

$$(a_{11} + ib_{11})(x_1 + iy_1) + (a_{12} + ib_{12})(x_2 + iy_2) = c_1 + id_1,$$
$$(a_{21} + ib_{21})(x_1 + iy_1) + (a_{22} + ib_{22})(x_2 + iy_2) = c_2 + id_2.$$

a. Use the properties of complex numbers to convert this system to the equivalent 4×4 real linear system

$$\begin{aligned} \text{Real part:} \quad & A\mathbf{x} - B\mathbf{y} = \mathbf{c}, \\ \text{Imaginary part:} \quad & B\mathbf{x} + A\mathbf{y} = \mathbf{d}. \end{aligned}$$

b. Solve the linear system

$$\begin{aligned} (1-2i)(x_1 + iy_1) + (3+2i)(x_2 + iy_2) &= 5 + 2i, \\ (2+i)(x_1 + iy_1) + (4+3i)(x_2 + iy_2) &= 4 - i. \end{aligned}$$

6.5 Matrix Factorization

Matrix factorization is another of the important techniques that Gauss seems to be the first to have discovered. It is included in his two-volume treatise on celestial mechanics *Theoria motus corporum coelestium in sectionibus conicis Solem ambientium*, which was published in 1809.

Gaussian elimination is the principal tool in the direct solution of linear systems of equations, so it should be no surprise that it appears in other guises. In this section we will see that the steps used to solve a system of the form $A\mathbf{x} = \mathbf{b}$ can be used to factor a matrix. The factorization is particularly useful when it has the form $A = LU$, where L is lower triangular and U is upper triangular. Although not all matrices have this type of representation, many do that occur frequently in the application of numerical techniques.

In Section 6.2 we found that Gaussian elimination applied to an arbitrary linear system $A\mathbf{x} = \mathbf{b}$ requires $O(n^3/3)$ arithmetic operations to determine $\mathbf{x}$. However, to solve a linear system that involves an upper-triangular system requires only backward substitution, which takes $O(n^2)$ operations. The number of operations required to solve a lower-triangular system is similar.

Suppose that A has been factored into the triangular form $A = LU$, where L is lower triangular and U is upper triangular. Then we can easily solve for $\mathbf{x}$ using a two-step process.

- First define the temporary vector $\mathbf{y} = U\mathbf{x}$ and solve the lower triangular system $L\mathbf{y} = \mathbf{b}$ for $\mathbf{y}$. Since L is triangular, determining $\mathbf{y}$ from this equation requires only $O(n^2)$ operations.
- Once $\mathbf{y}$ is known, the upper triangular system $U\mathbf{x} = \mathbf{y}$ requires only an additional $O(n^2)$ operations to determine the solution $\mathbf{x}$.

Solving a linear system $A\mathbf{x} = \mathbf{b}$ in factored form means that the number of operations needed to solve the system $A\mathbf{x} = \mathbf{b}$ is reduced from $O(n^3/3)$ to $O(2n^2)$.

Example 1 Compare the approximate number of operations required to determine the solution to a linear system using a technique requiring $O(n^3/3)$ operations and one requiring $O(2n^2)$ when $n = 20$, $n = 100$, and $n = 1000$.

Solution Table 6.3 gives the results of these calculations. ∎

Table 6.3

n	$n^3/3$	$2n^2$	Reduction
10	$3.\overline{3} \times 10^2$	2×10^2	40%
100	$3.\overline{3} \times 10^5$	2×10^4	94%
1000	$3.\overline{3} \times 10^8$	2×10^6	99.4%

As the example illustrates, the reduction factor increases dramatically with the size of the matrix. Not surprisingly, the reductions from the factorization come at a cost; determining the specific matrices L and U requires $O(n^3/3)$ operations. But once the factorization is determined, systems involving the matrix A can be solved in this simplified manner for any number of vectors $\mathbf{b}$.

To obtain the LU factorization of an $n \times n$ matrix A:

- use Gaussian elimination to solve a linear system of the form $A\mathbf{x} = \mathbf{b}$.
- if Gaussian elimination can be performed without row interchanges, then
  - the upper triangular matrix U is the matrix that results when Gaussian elimination is complete,
  - the lower triangular matrix L has 1s on its main diagonal, and each entry below the main diagonal is the multiplier that was needed to place a zero in that entry when Gaussian elimination was performed.

The process is outlined in the following example.

Example 2 Determine the LU factorization for matrix

$$A = \begin{bmatrix} 1 & 1 & 0 & 3 \\ 2 & 1 & -1 & 1 \\ 3 & -1 & -1 & 2 \\ -1 & 2 & 3 & -1 \end{bmatrix}$$

Solution A system involving the matrix A system was considered in the Illustration of Section 6.2 (see page 230), where we saw that the sequence of operations

$$(E_2 - 2E_1) \to (E_2), \quad (E_3 - 3E_1) \to (E_3), \quad (E_4 - (-1)E_1) \to (E_4,)$$

followed by

$$(E_3 - 4E_2) \to (E_3) \quad \text{and} \quad (E_4 - (-3)E_2) \to (E_4)$$

converts the system to the triangular system

$$\begin{aligned} x_1 + x_2 \qquad\quad + 3x_4 &= 4, \\ -x_2 - x_3 - 5x_4 &= -7, \\ 3x_3 + 13x_4 &= 13, \\ -13x_4 &= -13. \end{aligned}$$

As a consequence, the upper triangular matrix in the factorization is

$$U = \begin{bmatrix} 1 & 1 & 0 & 3 \\ 0 & -1 & -1 & -5 \\ 0 & 0 & 3 & 13 \\ 0 & 0 & 0 & -13 \end{bmatrix}.$$

The multipliers used in Gaussian elimination were

$$m_{21} = 2, \quad m_{31} = 3, \quad m_{41} = -1, \quad m_{32} = 4, \quad m_{42} = -3, \quad \text{and} \quad m_{43} = 0,$$

so the lower triangular matrix is

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 4 & 1 & 0 \\ -1 & -3 & 0 & 1 \end{bmatrix}$$

Hence the LU factorization of A is

$$A = \begin{bmatrix} 1 & 1 & 0 & 3 \\ 2 & 1 & -1 & 1 \\ 3 & -1 & -1 & 2 \\ -1 & 2 & 3 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 4 & 1 & 0 \\ -1 & -3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 3 \\ 0 & -1 & -1 & -5 \\ 0 & 0 & 3 & 13 \\ 0 & 0 & 0 & -13 \end{bmatrix} = LU.$$

■

The next example uses the factorization in the previous to solve a linear system.

Example 3 Use the factorization found in Example 2 to solve the system

$$\begin{aligned} x_1 + x_2 \qquad + 3x_4 &= 8, \\ 2x_1 + x_2 - x_3 + x_4 &= 7, \\ 3x_1 - x_2 - x_3 + 2x_4 &= 14, \\ -x_1 + 2x_2 + 3x_3 - x_4 &= -7. \end{aligned}$$

Solution To solve

$$A\mathbf{x} = LU\mathbf{x} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 4 & 1 & 0 \\ -1 & -3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 3 \\ 0 & -1 & -1 & -5 \\ 0 & 0 & 3 & 13 \\ 0 & 0 & 0 & -13 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 8 \\ 7 \\ 14 \\ -7 \end{bmatrix},$$

we first introduce the temporary vector $\mathbf{y} = U\mathbf{x}$. Then $\mathbf{b} = L(U\mathbf{x}) = L\mathbf{y}$. That is,

$$L\mathbf{y} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 4 & 1 & 0 \\ -1 & -3 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 8 \\ 7 \\ 14 \\ -7 \end{bmatrix}.$$

This system is solved for $\mathbf{y}$ by a simple forward-substitution process:

$$\begin{aligned} y_1 &= 8; \\ 2y_1 + y_2 &= 7, \quad \text{so } y_2 = 7 - 2y_1 = -9; \\ 3y_1 + 4y_2 + y_3 &= 14, \quad \text{so } y_3 = 14 - 3y_1 - 4y_2 = 26; \\ -y_1 - 3y_2 + y_4 &= -7, \quad \text{so } y_4 = -7 + y_1 + 3y_2 = -26. \end{aligned}$$

We then solve $U\mathbf{x} = \mathbf{y}$ for $\mathbf{x}$, the solution of the original system; that is,

$$\begin{bmatrix} 1 & 1 & 0 & 3 \\ 0 & -1 & -1 & -5 \\ 0 & 0 & 3 & 13 \\ 0 & 0 & 0 & -13 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 8 \\ -9 \\ 26 \\ -26 \end{bmatrix}.$$

Using backward substitution we obtain $x_4 = 2, x_3 = 0, x_2 = -1, x_1 = 3$. ■

The program LUFACT64 performs the LU Factorization.

Although new matrices $L = [l_{ij}]$ and $U = [u_{ij}]$ are constructed by the program LUFACT64, the values generated replace the corresponding entries of A that are no longer needed. Thus, the new matrix has entries $a_{ij} = l_{ij}$ for each $i = 2, 3, \ldots, n$ and $j = 1, 2, \ldots, i-1$ and $a_{ij} = u_{ij}$ for each $i = 1, 2, \ldots, n$ and $j = i+1, i+2, \ldots, n$.

The factorization is particularly useful when a number of linear systems involving A must be solved because most of the operations, those involving the Gaussian Elimination, need to be performed only once.

Permutation Matrices

In the previous discussion we assumed that A is such that a linear system of the form $A\mathbf{x} = \mathbf{b}$ can be solved using Gaussian elimination that does not require row interchanges. From a practical standpoint, this factorization is useful only when row interchanges are not required to control the round-off error resulting from the use of finite-digit arithmetic. Although many systems we encounter when using approximation methods are of this type, factorization modifications must be made when row interchanges are required. We begin the discussion with the introduction of a class of matrices that are used to rearrange, or permute, rows of a given matrix.

An $n \times n$ **permutation matrix** P has precisely one entry in each column and each row whose value is 1, and all of whose other entries are 0.

Illustration The matrix

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

is a 3×3 permutation matrix. For any 3×3 matrix A, multiplying on the left by P has the effect of interchanging the second and third rows of A:

$$PA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}.$$

Similarly, multiplying A on the right by P interchanges the second and third columns of A. □

The matrix multiplication PA permutes rows of A.

The matrix multiplication AP permutes columns of A.

There are two useful properties of permutation matrices that relate to Gaussian elimination. The first of these was shown in the Illustration.

- If $k_1, \ldots, k_n$ is a permutation of the integers $1, \ldots, n$ and the permutation matrix $P = [p_{ij}]$ is defined by

$$p_{ij} = \begin{cases} 1, & \text{if } j = k_i, \\ 0, & \text{otherwise,} \end{cases} \quad \text{then} \quad PA = \begin{bmatrix} a_{k_1,1} & a_{k_1,2} & \cdots & a_{k_1,n} \\ a_{k_2,1} & a_{k_2,2} & \cdots & a_{k_2,n} \\ \vdots & \vdots & & \vdots \\ a_{k_n,1} & a_{k_n,2} & \cdots & a_{k_n,n} \end{bmatrix}.$$

The second is

- If P is a permutation matrix, then P^{-1} exists and $P^{-1} = P^t$.

Example 4 Determine a factorization in the form $A = (P^tL)U$ for the matrix

$$A = \begin{bmatrix} 0 & 0 & -1 & 1 \\ 1 & 1 & -1 & 2 \\ -1 & -1 & 2 & 0 \\ 1 & 2 & 0 & 2 \end{bmatrix}.$$

Solution The matrix A does not have an LU factorization because $a_{11} = 0$. However, using the row interchange $(E_1) \leftrightarrow (E_2)$, followed by $(E_3 + E_1) \rightarrow (E_3)$ and $(E_4 - E_1) \rightarrow (E_4)$, produces

$$\begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 1 & 1 & 0 \end{bmatrix}.$$

Then the row interchange $(E_2) \leftrightarrow (E_4)$, followed by $(E_4 + E_3) \rightarrow (E_4)$, gives the matrix

$$U = \begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 3 \end{bmatrix}.$$

The permutation matrix associated with the row interchanges $(E_1) \leftrightarrow (E_2)$ and $(E_2) \leftrightarrow (E_4)$ is

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix},$$

and

$$PA = \begin{bmatrix} 1 & 1 & -1 & 2 \\ 1 & 2 & 0 & 2 \\ -1 & -1 & 2 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}.$$

Gaussian elimination is performed on PA using the same operations as on A, except without the row interchanges. That is, $(E_2 - E_1) \rightarrow (E_2)$, $(E_3 + E_1) \rightarrow (E_3)$, followed by $(E_4 + E_3) \rightarrow (E_4)$. The nonzero multipliers for PA are, consequently,

$$m_{21} = 1, \quad m_{31} = -1, \quad \text{and} \quad m_{43} = -1,$$

and the LU factorization of PA is

$$PA = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 3 \end{bmatrix} = LU.$$

Multiplying by $P^{-1} = P^t$ produces the factorization

$$A = P^{-1}(LU) = P^t(LU) = (P^tL)U = \begin{bmatrix} 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 3 \end{bmatrix}.$$

■

MATLAB has the command `lu(A)` to obtain an LU factorization of a matrix A in the form $A = PLU$, where U is upper triangular, L is lower triangular, and P is a permutation matrix. Notice that the permutation matrix P that MATLAB constructs is the matrix that we would call P^t. We apply this command to the matrix in Example 4 by first defining

```
A = [0 0 -1 1; 1 1 -1 2; -1 -1 1 0; 1 2 0 2]
```

and then calling

```
[L, U, P] = lu(A)
```

MATLAB responds with

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 3 \end{bmatrix}, \text{ and } P = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

For these matrices L, U, and P, we have $A = PLU$.

EXERCISE SET 6.5

1. Solve the following linear systems.

   a. $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 & -1 \\ 0 & -2 & 1 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$

   b. $\begin{bmatrix} 2 & 0 & 0 \\ -1 & 1 & 0 \\ 3 & 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix}$

2. Factor the following matrices into the LU decomposition with $l_{ii} = 1$ for all i.

   a. $\begin{bmatrix} 2 & -1 & 1 \\ 3 & 3 & 9 \\ 3 & 3 & 5 \end{bmatrix}$

   b. $\begin{bmatrix} 1.012 & -2.132 & 3.104 \\ -2.132 & 4.096 & -7.013 \\ 3.104 & -7.013 & 0.014 \end{bmatrix}$

   c. $\begin{bmatrix} 2 & 0 & 0 & 0 \\ 1 & 1.5 & 0 & 0 \\ 0 & -3 & 0.5 & 0 \\ 2 & -2 & 1 & 1 \end{bmatrix}$

   d. $\begin{bmatrix} 2.1756 & 4.0231 & -2.1732 & 5.1967 \\ -4.0231 & 6.0000 & 0 & 1.1973 \\ -1.0000 & -5.2107 & 1.1111 & 0 \\ 6.0235 & 7.0000 & 0 & -4.1561 \end{bmatrix}$

3. Obtain factorizations of the form $A = P^t LU$ for the following matrices.

   a. $A = \begin{bmatrix} 0 & 2 & 3 \\ 1 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}$

   b. $A = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 2 & 3 \\ 2 & -1 & 4 \end{bmatrix}$

   c. $A = \begin{bmatrix} 1 & -2 & 3 & 0 \\ 3 & -6 & 9 & 3 \\ 2 & 1 & 4 & 1 \\ 1 & -2 & 2 & -2 \end{bmatrix}$

   d. $A = \begin{bmatrix} 1 & -2 & 3 & 0 \\ 1 & -2 & 3 & 1 \\ 1 & -2 & 2 & -2 \\ 2 & 1 & 3 & -1 \end{bmatrix}$

4. Suppose $A = P^tLU$, where P is a permutation matrix, L is a lower-triangular matrix with 1s on the diagonal, and U is an upper-triangular matrix.
   a. Count the number of operations needed to compute P^tLU for a given matrix A.
   b. Show that if P contains k row interchanges, then
   $$\det P = \det P^t = (-1)^k.$$
   c. Use $\det A = \det P^t \cdot \det L \cdot \det U = (-1)^k \det U$ to count the number of operations for determining $\det A$ by factoring.
   d. Factor A as P^tLU and use this factorization to compute $\det A$ and to count the number of operations when
   $$A = \begin{bmatrix} 0 & 2 & 1 & 4 & -1 & 3 \\ 1 & 2 & -1 & 3 & 4 & 0 \\ 0 & 1 & 1 & -1 & 2 & -1 \\ 2 & 3 & -4 & 2 & 0 & 5 \\ 1 & 1 & 1 & 3 & 0 & 2 \\ -1 & -1 & 2 & -1 & 2 & 0 \end{bmatrix}.$$

5. Use the LU factorization obtained in Exercise 2 to solve the following linear systems.
   a. $$\begin{aligned} 2x_1 - x_2 + x_3 &= -1, \\ 3x_1 + 3x_2 + 9x_3 &= 0, \\ 3x_1 + 3x_2 + 5x_3 &= 4. \end{aligned}$$
   b. $$\begin{aligned} 1.012x_1 - 2.132x_2 + 3.104x_3 &= 1.984, \\ -2.132x_1 + 4.096x_2 - 7.013x_3 &= -5.049, \\ 3.104x_1 - 7.013x_2 + 0.014x_3 &= -3.895. \end{aligned}$$
   c. $$\begin{aligned} 2x_1 \qquad\qquad\qquad &= 3, \\ x_1 + 1.5x_2 \qquad\quad &= 4.5, \\ - 3x_2 + 0.5x_3 \quad &= -6.6, \\ 2x_1 - 2x_2 + x_3 + x_4 &= 0.8. \end{aligned}$$
   d. $$\begin{aligned} 2.1756x_1 + 4.0231x_2 - 2.1732x_3 + 5.1967x_4 &= 17.102, \\ -4.0231x_1 + 6.0000x_2 \qquad\qquad + 1.1973x_4 &= -6.1593, \\ -1.0000x_1 - 5.2107x_2 + 1.1111x_3 \qquad\qquad &= 3.0004, \\ 6.0235x_1 + 7.0000x_2 \qquad\qquad - 4.1561x_4 &= 0.0000. \end{aligned}$$

6.6 Techniques for Special Matrices

Although this chapter has been concerned primarily with the effective application of Gaussian elimination for finding the solution to a linear system of equations, many of the results have wider application. It might be said that Gaussian elimination is the hub about which the chapter revolves, but the wheel itself is of equal interest and has application in many forms in the study of numerical methods. In this section we consider some matrices that are of special types, forms that will be used in other chapters of the book.

Strict Diagonal Dominance

Each main diagonal entry in a strictly diagonally dominant matrix has a magnitude that is strictly greater than the sum of the magnitudes of all the other entries in that row.

The $n \times n$ matrix A is said to be **strictly diagonally dominant** when

$$|a_{ii}| > \sum_{\substack{j=1,\\ j\neq i}}^{n} |a_{ij}| \quad \text{holds for each } i = 1, 2, \ldots, n.$$

Illustration Consider the matrices

$$A = \begin{bmatrix} 7 & 2 & 0 \\ 3 & 5 & -1 \\ 0 & 5 & -6 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 6 & 4 & -3 \\ 4 & -2 & 0 \\ -3 & 0 & 1 \end{bmatrix}.$$

The nonsymmetric matrix A is strictly diagonally dominant because

$$|7| > |2| + |0|, \quad |5| > |3| + |-1|, \quad \text{and} \quad |-6| > |0| + |5|.$$

The symmetric matrix B is not strictly diagonally dominant because, for example, in the first row the absolute value of the diagonal element is $|6| < |4| + |-3| = 7$. It is interesting to note that A^t is not strictly diagonally dominant because the middle row of A^t is [2 5 5], nor, of course, is B^t because $B^t = B$. □

Strictly Diagonally Dominant Matrices

A strictly diagonally dominant matrix A has an inverse. Moreover, Gaussian elimination can be performed on any linear system of the form $A\mathbf{x} = \mathbf{b}$ to obtain its unique solution without row or column interchanges, and the computations are stable with respect to the growth of round-off error.

Positive Definite Matrices

The name positive definite refers to the fact that the number $\mathbf{x}^t A\mathbf{x}$ must be positive whenever $\mathbf{x} \neq \mathbf{0}$.

A matrix A is **positive definite** if it is symmetric and if $\mathbf{x}^t A\mathbf{x} > 0$ for every n-dimensional column vector $\mathbf{x} \neq \mathbf{0}$.

Using the definition to determine whether a matrix is positive definite can be difficult. Fortunately, there are more easily verified criteria for identifying members that are and are not of this important class.

Positive Definite Matrix Properties

If A is an $n \times n$ positive definite matrix, then

(i) A has an inverse;

(ii) $a_{ii} > 0$ for each $i = 1, 2, \ldots, n$;

(iii) $\max_{1\le k,j\le n} |a_{kj}| \le \max_{1\le i\le n} |a_{ii}|$;

(iv) $(a_{ij})^2 < a_{ii}a_{jj}$ for each $i \neq j$.

Our definition of positive definite requires the matrix to be symmetric, but not all authors make this requirement. For example, Golub and Van Loan [GV], a standard reference in matrix methods, requires only that $\mathbf{x}^t A\mathbf{x} > 0$ for each nonzero vector $\mathbf{x}$. Matrices that we call positive definite are called symmetric positive definite in [GV]. Keep this discrepancy in mind if you are using material from other sources.

The next result parallels the strictly diagonally dominant result presented previously.

Positive Definite Matrix Equivalences

The following are equivalent for any $n \times n$ symmetric matrix A:

(i) A is positive definite.

(ii) Gaussian elimination without row interchanges can be performed on the linear system $A\mathbf{x} = \mathbf{b}$ with all pivot elements positive. (This ensures that the computations are stable with respect to the growth of round-off error.)

(iii) A can be factored in the form LL^t, where L is lower triangular with positive diagonal entries.

(iv) A can be factored in the form LDL^t, where L is lower triangular with 1s on its diagonal and D is a diagonal matrix with positive diagonal entries.

(v) For each $i = 1, 2, \ldots, n$, we have

$$\det \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1i} \\ a_{21} & a_{22} & \cdots & a_{2i} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ii} \end{bmatrix} > 0.$$

The next examples illustrate portions of this result. First we will consider (v).

Example 1 Show that the symmetric matrix

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

is positive definite.

Solution We have

$$\det[2] = 2 > 0, \quad \det \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = 4 - 1 = 3 > 0,$$

and

$$\det \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} = 2 \det \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} - (-1) \det \begin{bmatrix} -1 & -1 \\ 0 & 2 \end{bmatrix}$$

$$= 2(4 - 1) + (-2 + 0) = 4 > 0,$$

so, by (v), A is positive definite. ■

The program LDLFCT65 performs the LDL^t Factorization.

The next example illustrates how the LDL^t factorization of a positive definite matrix described in (iv) of the result is formed.

Example 2 Determine the LDL^t factorization of the positive definite matrix

$$A = \begin{bmatrix} 4 & -1 & 1 \\ -1 & 4.25 & 2.75 \\ 1 & 2.75 & 3.5 \end{bmatrix}.$$

Solution The LDL^t factorization has 1s on the diagonal of the lower triangular matrix L so we need to have

$$A = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{21} & a_{22} & a_{32} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix} \begin{bmatrix} 1 & l_{21} & l_{31} \\ 0 & 1 & l_{32} \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} d_1 & d_1 l_{21} & d_1 l_{31} \\ d_1 l_{21} & d_2 + d_1 l_{21}^2 & d_2 l_{32} + d_1 l_{21} l_{31} \\ d_1 l_{31} & d_1 l_{21} l_{31} + d_2 l_{32} & d_1 l_{31}^2 + d_2 l_{32}^2 + d_3 \end{bmatrix}$$

Thus

$$a_{11}:\ 4 = d_1 \implies d_1 = 4, \qquad a_{21}:\ -1 = d_1 l_{21} \implies l_{21} = -0.25$$

$$a_{31}:\ 1 = d_1 l_{31} \implies l_{31} = 0.25, \qquad a_{22}:\ 4.25 = d_2 + d_1 l_{21}^2 \implies d_2 = 4$$

$$a_{32}:\ 2.75 = d_1 l_{21} l_{31} + d_2 l_{32} \implies l_{32} = 0.75, \quad a_{33}:\ 3.5 = d_1 l_{31}^2 + d_2 l_{32}^2 + d_3 \implies d_3 = 1,$$

and we have

$$A = LDL^t = \begin{bmatrix} 1 & 0 & 0 \\ -0.25 & 1 & 0 \\ 0.25 & 0.75 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -0.25 & 0.25 \\ 0 & 1 & 0.75 \\ 0 & 0 & 1 \end{bmatrix}. \qquad \blacksquare$$

Andre-Louis Cholesky (1875–1918) was a French military officer involved in geodesy and surveying in the early 1900s. He developed this factorization method to compute solutions to least squares problems.

Any symmetric matrix A for which Gaussian elimination can be applied without row interchanges can be factored into the form LDL^t. In this general case, L is lower triangular with 1s on its diagonal, and D is the diagonal matrix with the Gaussian elimination pivots on its diagonal. This result is widely applied because symmetric matrices are common and easily recognized.

The factorization in part (iii) of the positive definite matrix equivalences, that is, $A = LL^t$, is known as Cholesky's factorization. The next example shows how this is done.

Example 3 Determine the Cholesky LL^t factorization of the positive definite matrix

$$A = \begin{bmatrix} 4 & -1 & 1 \\ -1 & 4.25 & 2.75 \\ 1 & 2.75 & 3.5 \end{bmatrix}.$$

The program CHOLFC66 performs the LL^t Factorization.

Solution The LL^t factorization does not necessarily have 1s on the diagonal of the lower triangular matrix L so we need to have

$$A = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{21} & a_{22} & a_{32} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} l_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{32} \\ 0 & 0 & l_{33} \end{bmatrix}$$

$$= \begin{bmatrix} l_{11}^2 & l_{11} l_{21} & l_{11} l_{31} \\ l_{11} l_{21} & l_{21}^2 + l_{22}^2 & l_{21} l_{31} + l_{22} l_{32} \\ l_{11} l_{31} & l_{21} l_{31} + l_{22} l_{32} & l_{31}^2 + l_{32}^2 + l_{33}^2 \end{bmatrix}$$

Thus

$$a_{11}:\ 4 = l_{11}^2 \implies l_{11} = 2, \qquad a_{21}:\ -1 = l_{11} l_{21} \implies l_{21} = -0.5$$

$$a_{31}:\ 1 = l_{11} l_{31} \implies l_{31} = 0.5, \qquad a_{22}:\ 4.25 = l_{21}^2 + l_{22}^2 \implies l_{22} = 2$$

$$a_{32}:\ 2.75 = l_{21} l_{31} + l_{22} l_{32} \implies l_{32} = 1.5, \quad a_{33}:\ 3.5 = l_{31}^2 + l_{32}^2 + l_{33}^2 \implies l_{33} = 1,$$

and we have

$$A = LL^t = \begin{bmatrix} 2 & 0 & 0 \\ -0.5 & 2 & 0 \\ 0.5 & 1.5 & 1 \end{bmatrix} \begin{bmatrix} 2 & -0.5 & 0.5 \\ 0 & 2 & 1.5 \\ 0 & 0 & 1 \end{bmatrix}.$$

■

MATLAB commands are available for computing both the LDL^t (`ldl`) and Cholesky (LL^t) (`chol`) factorizations. For example, for the matrix A defined by

```
A = [4 -1 1; -1 4.25 2.75; 1 2.75 3.5]
```

we have

```
[L, D] = ldl(A)
```

giving

$$L = \begin{bmatrix} 1.000000000000000 & 0 & 0 \\ -0.250000000000000 & 1.000000000000000 & 0 \\ 0.250000000000000 & 0.750000000000000 & 1.000000000000000 \end{bmatrix} \quad \text{and}$$

$$D = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and

```
L=chol(A)
```

giving

$$L = \begin{bmatrix} 2.000000000000000 & -0.500000000000000 & 0.500000000000000 \\ 0 & 2.000000000000000 & 1.500000000000000 \\ 0 & 0 & 1.000000000000000 \end{bmatrix}$$

Band Matrices

The last special matrices considered are *band matrices*. In many applications, band matrices are also strictly diagonally dominant or positive definite. This combination of properties is very useful.

The name for a band matrix comes from the fact that all the nonzero entries lie in a band that is centered on the main diagonal.

An $n \times n$ matrix is called a **band matrix** if integers p and q, with $1 < p, q < n$, exist with the property that $a_{ij} = 0$ whenever $p \le j - i$ or $q \le i - j$. The number p describes the number of diagonals above, and including, the main diagonal on which nonzero entries may lie. The number q describes the number of diagonals below, and including, the main diagonal on which nonzero entries may lie. In most applications, $p = q$ and the nonzero entries are evenly banded about the main diagonal.

The **bandwidth** of the band matrix is $w = p + q - 1$, which tells us how many of the diagonals can contain nonzero entries. The 1 is subtracted from the sum of p and q because both of these numbers count the main diagonal.

For example, the matrix

$$A = \begin{bmatrix} 7 & 2 & 1 & 0 \\ 3 & 5 & -3 & -2 \\ 0 & 4 & 6 & -1 \\ 0 & 0 & 5 & 8 \end{bmatrix}$$

is a band matrix with $p = 3$ and $q = 2$, so it has bandwidth $3 + 2 - 1 = 4$.

Tridiagonal Matrices

Band matrices concentrate all their nonzero entries about the diagonal. Two special cases of band matrices that occur often have $p = q = 2$ and $p = q = 4$. Matrices of bandwidth 3 that occur when $p = q = 2$ are called **tridiagonal** because they have the form

$$A = \begin{bmatrix} a_{11} & a_{12} & 0 & \cdots & \cdots & 0 \\ a_{21} & a_{22} & a_{23} & \ddots & & \vdots \\ 0 & a_{32} & a_{33} & a_{34} & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & a_{n-1,n} \\ 0 & \cdots & \cdots & 0 & a_{n-1,n} & a_{nn} \end{bmatrix}.$$

Tridiagonal matrices will appear in Chapter 11 in connection with the study of piecewise linear approximations to boundary-value problems. The case of $p = q = 4$ will also be used in that chapter for the solution of boundary-value problems, when the approximating functions assume the form of cubic splines.

The program CRTRLS67 performs Crout Factorization.

The factorization methods can be simplified considerably in the case of band matrices because a large number of zeros appear in regular patterns. Of particular interest is the Crout factorization, where $A = LU$ with U having all 1s on its diagonal.

Crout factorization is illustrated in the following example.

Example 4 Determine the Crout factorization of the symmetric tridiagonal matrix

$$\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}.$$

Solution The LU factorization of A has the form

$$A = \begin{bmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 \\ 0 & a_{32} & a_{33} & a_{34} \\ 0 & 0 & a_{43} & a_{44} \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 & 0 \\ l_{21} & l_{22} & 0 & 0 \\ 0 & l_{32} & l_{33} & 0 \\ 0 & 0 & l_{43} & l_{44} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & 0 & 0 \\ 0 & 1 & u_{23} & 0 \\ 0 & 0 & 1 & u_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} l_{11} & l_{11}u_{12} & 0 & 0 \\ l_{21} & l_{22} + l_{21}u_{12} & l_{22}u_{23} & 0 \\ 0 & l_{32} & l_{33} + l_{32}u_{23} & l_{33}u_{34} \\ 0 & 0 & l_{43} & l_{44} + l_{43}u_{34} \end{bmatrix}.$$

Thus

$$\begin{aligned}
&a_{11}:\ 2 = l_{11} \implies l_{11} = 2, && a_{12}:\ -1 = l_{11}u_{12} \implies u_{12} = -\tfrac{1}{2}, \\
&a_{21}:\ -1 = l_{21} \implies l_{21} = -1, && a_{22}:\ 2 = l_{22} + l_{21}u_{12} \implies l_{22} = \tfrac{3}{2}, \\
&a_{23}:\ -1 = l_{22}u_{23} \implies u_{23} = -\tfrac{2}{3}, && a_{32}:\ -1 = l_{32} \implies l_{32} = -1, \\
&a_{33}:\ 2 = l_{33} + l_{32}u_{23} \implies l_{33} = \tfrac{4}{3}, && a_{34}:\ -1 = l_{33}u_{34} \implies u_{34} = -\tfrac{3}{4}, \\
&a_{43}:\ -1 = l_{43} \implies l_{43} = -1, && a_{44}:\ 2 = l_{44} + l_{43}u_{34} \implies l_{44} = \tfrac{5}{4}.
\end{aligned}$$

This gives the Crout factorization

$$A = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 & 0 \\ -1 & \frac{3}{2} & 0 & 0 \\ 0 & -1 & \frac{4}{3} & 0 \\ 0 & 0 & -1 & \frac{5}{4} \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{2} & 0 & 0 \\ 0 & 1 & -\frac{2}{3} & 0 \\ 0 & 0 & 1 & -\frac{3}{4} \\ 0 & 0 & 0 & 1 \end{bmatrix} = LU.$$

■

The next example shows how a linear system is solved once the Crout factorization is known.

Example 5 Use this Crout factorization found in Example 4 to solve the linear system

$$\begin{aligned} 2x_1 - x_2 \qquad\qquad &= 1, \\ -x_1 + 2x_2 - x_3 \qquad &= 0, \\ - x_2 + 2x_3 - x_4 &= 0, \\ - x_3 + 2x_4 &= 1. \end{aligned}$$

Solution First we introduce a temporary vector $\mathbf{y} = U\mathbf{x}$ and use forward substitution to solve the system

$$L\mathbf{y} = \begin{bmatrix} 2 & 0 & 0 & 0 \\ -1 & \frac{3}{2} & 0 & 0 \\ 0 & -1 & \frac{4}{3} & 0 \\ 0 & 0 & -1 & \frac{5}{4} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

This gives

$$\begin{aligned} 2y_1 = 1 &\implies y_1 = \frac{1}{2}, \\ -1y_1 + \frac{3}{2}y_2 = 0 &\implies y_2 = \frac{1}{3} \\ -y_2 + \frac{4}{3}y_3 = 0 &\implies y_3 = \frac{1}{4}, \\ -y_3 + \frac{5}{4}y_4 = 1 &\implies y_4 = 1, \end{aligned}$$

so $\mathbf{y} = \left(\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, 1\right)^t$.

Then using backward substitution to solve $U\mathbf{x} = \mathbf{y}$,

$$U\mathbf{x} = \begin{bmatrix} 1 & -\frac{1}{2} & 0 & 0 \\ 0 & 1 & -\frac{2}{3} & 0 \\ 0 & 0 & 1 & -\frac{3}{4} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{3} \\ \frac{1}{4} \\ 1 \end{bmatrix}$$

gives

$$x_4 = 1,$$
$$x_3 - \frac{3}{4}x_4 = \frac{1}{4} \implies x_3 = 1,$$
$$x_2 - \frac{2}{3}x_3 = \frac{1}{3} \implies x_2 = 1,$$
$$x_1 - \frac{1}{2}x_2 = \frac{1}{2} \implies x_1 = 1$$

and $\mathbf{x} = (1, 1, 1, 1)^t$. ■

The tridiagonal factorization can be applied whenever $l_i \neq 0$ for each $i = 1, 2, \ldots, n$. Two conditions, either of which ensure that this is true, are that the coefficient matrix of the system is positive definite or that it is strictly diagonally dominant. An additional condition that ensures this method can be applied is as follows.

Nonsingular Tridiagonal Matrices

Suppose that A is tridiagonal with $a_{i,i-1} \neq 0$ and $a_{i,i+1} \neq 0$ for each $i = 2, 3, \ldots, n-1$. If $|a_{11}| > |a_{12}|$, $|a_{nn}| > |a_{n,n-1}|$, and $|a_{ii}| \geq |a_{i,i-1}| + |a_{i,i+1}|$ for each $i = 2, 3, \ldots, n-1$, then A is nonsingular, and the values of l_i are nonzero for each $i = 1, 2, \ldots, n$.

EXERCISE SET 6.6

1. Determine which of the following matrices are (i) symmetric, (ii) singular, (iii) strictly diagonally dominant, and (iv) positive definite.

   a. $\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$ **b.** $\begin{bmatrix} -2 & 1 \\ 1 & -3 \end{bmatrix}$

   c. $\begin{bmatrix} 2 & 1 & 0 \\ 0 & 3 & 0 \\ 1 & 0 & 4 \end{bmatrix}$ **d.** $\begin{bmatrix} 2 & 1 & 0 \\ 0 & 3 & 2 \\ 1 & 2 & 4 \end{bmatrix}$

   e. $\begin{bmatrix} 4 & 2 & 6 \\ 3 & 0 & 7 \\ -2 & -1 & -3 \end{bmatrix}$ **f.** $\begin{bmatrix} 2 & -1 & 0 \\ -1 & 4 & 2 \\ 0 & 2 & 2 \end{bmatrix}$

   g. $\begin{bmatrix} 4 & 0 & 0 & 0 \\ 6 & 7 & 0 & 0 \\ 9 & 11 & 1 & 0 \\ 5 & 4 & 1 & 1 \end{bmatrix}$ **h.** $\begin{bmatrix} 2 & 3 & 1 & 2 \\ -2 & 4 & -1 & 5 \\ 3 & 7 & 1.5 & 1 \\ 6 & -9 & 3 & 7 \end{bmatrix}$

2. Find a factorizaton of the form $A = LDL^t$ for the following symmetric matrices:

   a. $A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$ **b.** $A = \begin{bmatrix} 4 & 1 & 1 & 1 \\ 1 & 3 & -1 & 1 \\ 1 & -1 & 2 & 0 \\ 1 & 1 & 0 & 2 \end{bmatrix}$

   c. $A = \begin{bmatrix} 4 & 1 & -1 & 0 \\ 1 & 3 & -1 & 0 \\ -1 & -1 & 5 & 2 \\ 0 & 0 & 2 & 4 \end{bmatrix}$ **d.** $A = \begin{bmatrix} 6 & 2 & 1 & -1 \\ 2 & 4 & 1 & 0 \\ 1 & 1 & 4 & -1 \\ -1 & 0 & -1 & 3 \end{bmatrix}$

3. Find a factorization of the form $A = LL^t$ for the matrices in Exercise 2.

4. Use the factorization in Exercise 2 to solve the following linear systems.

a. $$\begin{aligned} 2x_1 - x_2 \quad &= 3, \\ -x_1 + 2x_2 - x_3 &= -3, \\ -x_2 + 2x_3 &= 1. \end{aligned}$$

b. $$\begin{aligned} 4x_1 + x_2 + x_3 + x_4 &= 0.65, \\ x_1 + 3x_2 - x_3 + x_4 &= 0.05, \\ x_1 - x_2 + 2x_3 \quad &= 0, \\ x_1 + x_2 \quad + 2x_4 &= 0.5. \end{aligned}$$

c. $$\begin{aligned} 4x_1 + x_2 - x_3 \quad &= 7, \\ x_1 + 3x_2 - x_3 \quad &= 8, \\ -x_1 - x_2 + 5x_3 + 2x_4 &= -4, \\ 2x_3 + 4x_4 &= 6. \end{aligned}$$

d. $$\begin{aligned} 6x_1 + 2x_2 + x_3 - x_4 &= 0, \\ 2x_1 + 4x_2 + x_3 \quad &= 7, \\ x_1 + x_2 + 4x_3 - x_4 &= -1, \\ -x_1 \quad - x_3 + 3x_4 &= -2. \end{aligned}$$

5. Use Crout factorization for tridiagonal systems to solve the following linear systems.

a. $$\begin{aligned} x_1 - x_2 \quad &= 0, \\ -2x_1 + 4x_2 - 2x_3 &= -1, \\ -x_2 + 2x_3 &= 1.5. \end{aligned}$$

b. $$\begin{aligned} 3x_1 + x_2 \quad &= -1, \\ 2x_1 + 4x_2 + x_3 &= 7, \\ 2x_2 + 5x_3 &= 9. \end{aligned}$$

c. $$\begin{aligned} 2x_1 - x_2 \quad &= 3, \\ -x_1 + 2x_2 - x_3 &= -3, \\ -x_2 + 2x_3 &= 1. \end{aligned}$$

d. $$\begin{aligned} 0.5x_1 + 0.25x_2 \quad &= 0.35, \\ 0.35x_1 + 0.8x_2 + 0.4x_3 \quad &= 0.77, \\ 0.25x_2 + x_3 + 0.5x_4 &= -0.5, \\ x_3 - 2x_4 &= -2.25. \end{aligned}$$

6. Let A be the 10×10 tridiagonal matrix given by $a_{ii} = 2$, $a_{i,i+1} = a_{i,i-1} = -1$, for each $i = 2, \ldots, 9$, and $a_{11} = a_{10,10} = 2$, $a_{12} = a_{10,9} = -1$. Let $\mathbf{b}$ be the 10-dimensional column vector given by $b_1 = b_{10} = 1$ and $b_i = 0$ for each $i = 2, 3, \ldots, 9$. Solve $A\mathbf{x} = \mathbf{b}$ using the Crout factorization for tridiagonal systems.

7. Suppose that A and B are positive definite $n \times n$ matrices.

   a. Must $-A$ also be positive definite?
   b. Must A^t also be positive definite?
   c. Must $A + B$ also be positive definite?
   d. Must A^2 also be positive definite?
   e. Must $A - B$ also be positive definite?

8. Let

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ -1 & 1 & \alpha \end{bmatrix}.$$

Find all values of α for which

   a. A is singular.
   b. A is strictly diagonally dominant.
   c. A is symmetric.
   d. A is positive definite.

9. Let

$$A = \begin{bmatrix} \alpha & 1 & 0 \\ \beta & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}.$$

Find all values of α and β for which

   a. A is singular.
   b. A is strictly diagonally dominant.

c. A is symmetric.

d. A is positive definite.

10. Suppose A and B commute; that is, $AB = BA$. Must A^t and B^t also commute?

11. In a paper by Dorn and Burdick [DoB], it is reported that the average wing length that resulted from mating three mutant varieties of fruit flies (*Drosophila melanogaster*) can be expressed in the symmetric matrix form

$$A = \begin{bmatrix} 1.59 & 1.69 & 2.13 \\ 1.69 & 1.31 & 1.72 \\ 2.13 & 1.72 & 1.85 \end{bmatrix},$$

where a_{ij} denotes the average wing length of an offspring resulting from the mating of a male of type i with a female of type j.

a. What physical significance is associated with the symmetry of this matrix?

b. Is this matrix positive definite? If so, prove it; if not, find a nonzero vector $\mathbf{x}$ for which $\mathbf{x}^t A\mathbf{x} \leq 0$.

6.7 Survey of Methods and Software

In this chapter we have looked at direct methods for solving linear systems. A linear system consists of n equations in n unknowns expressed in matrix notation as $A\mathbf{x} = \mathbf{b}$. These techniques use a finite sequence of arithmetic operations to determine the exact solution of the system subject only to round-off error. We found that the linear system $A\mathbf{x} = \mathbf{b}$ has a unique solution if and only if A^{-1} exists, which is equivalent to $\det A \neq 0$. The solution of the linear system is the vector $\mathbf{x} = A^{-1}\mathbf{b}$.

Pivoting techniques were introduced to minimize the effects of round-off error, which can dominate the solution when using direct methods. We studied partial pivoting, scaled partial pivoting, and total pivoting. We recommend the partial or scaled partial pivoting methods for most problems because these decrease the effects of round-off error without adding much extra computation. Total pivoting should be used if round-off error is suspected to be large. In Section 7.6 we will see some procedures for estimating this round-off error.

Gaussian elimination was shown to yield a factorization of the matrix A into LU, where L is lower triangular with 1s on the diagonal and U is upper triangular. (This process is sometimes called Doolittle's factorization.) Not all nonsingular matrices can be factored this way, but a permutation of the rows will always give a factorization of the form $PA = LU$, where P is the permutation matrix used to rearrange the rows of A. The advantage of the factorization is that the work is reduced when solving linear systems $A\mathbf{x} = \mathbf{b}$ with the same coefficient matrix A and different vectors $\mathbf{b}$.

Factorizations take a simpler form when the matrix A is positive definite. For example, the Cholesky factorization has the form $A = LL^t$, where L is lower triangular. A symmetric matrix that has an LU factorization can also be factored in the form $A = LDL^t$, where D is diagonal and L is lower triangular with 1s on the diagonal. With these factorizations, manipulations involving A can be simplified. If A is tridiagonal, the LU factorization takes a particularly simple form, with U having 1s on the main diagonal and its only other nonzero entries on the diagonal immediately above. In addition, L has its only nonzero entries on the main diagonal and on the diagonal immediately below. Another important matrix factorization technique is the singular value decomposition considered in Section 9.6.

The direct methods are the methods of choice for most linear systems. For tridiagonal, banded, and positive definite matrices, the special methods are recommended. For the general case, Gaussian elimination or LU factorization methods, which allow pivoting, are recommended. In these cases, the effects of round-off error should be monitored. In Section 7.6 we discuss estimating errors in direct methods.