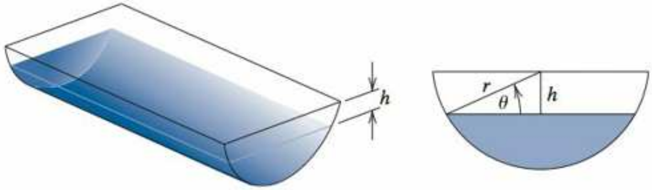


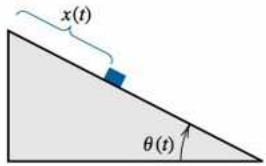
17. The particle in the figure starts at rest on a smooth inclined plane whose angle θ is changing at a constant rate

$$\frac{d\theta}{dt} = \omega < 0.$$

At the end of t seconds, the position of the object is given by

$$x(t) = \frac{g}{2\omega^2}\left(\frac{e^{\omega t} - e^{-\omega t}}{2} - \sin \omega t\right).$$

Suppose the particle has moved 1.7 ft in 1 s. Find, to within 10^{-5}, the rate ω at which θ changes. Assume that $g = -32.17$ ft/s^2.



2.4 Newton's Method

Isaac Newton (1641–1727) was one of the most brilliant scientists of all time. The late 17th century was a vibrant period for science and mathematics and Newton's work touches nearly every aspect of mathematics. His method for solving was introduced to find a root of the equation $y^3 - 2y - 5 = 0$. Although he demonstrated the method only for polynomials, it is clear that he realized its broader applications.

The Bisection and Secant methods both have geometric representations that use the zero of an approximating line to the graph of a function f to approximate the solution to $f(x) = 0$. The increase in accuracy of the Secant method over the Bisection method is a consequence of the fact that the secant line to the curve better approximates the graph of f than does the line used to generate the approximations in the Bisection method.

The line that *best* approximates the graph of the function at a point on its graph is the tangent line to the graph at that point. Using this line instead of the secant line produces **Newton's method** (also called the *Newton–Raphson method*), the technique we consider in this section.

Newton's Method

Suppose that p_0 is an initial approximation to the root p of the equation $f(x) = 0$ and that f' exists in an interval containing all the approximations to p. The slope of the tangent line to the graph of f at the point $(p_0, f(p_0))$ is $f'(p_0)$, so the equation of this tangent line is

$$y - f(p_0) = f'(p_0)(x - p_0).$$

This tangent line crosses the x-axis when the y-coordinate of the point on the line is 0, so the next approximation, p_1, to p satisfies

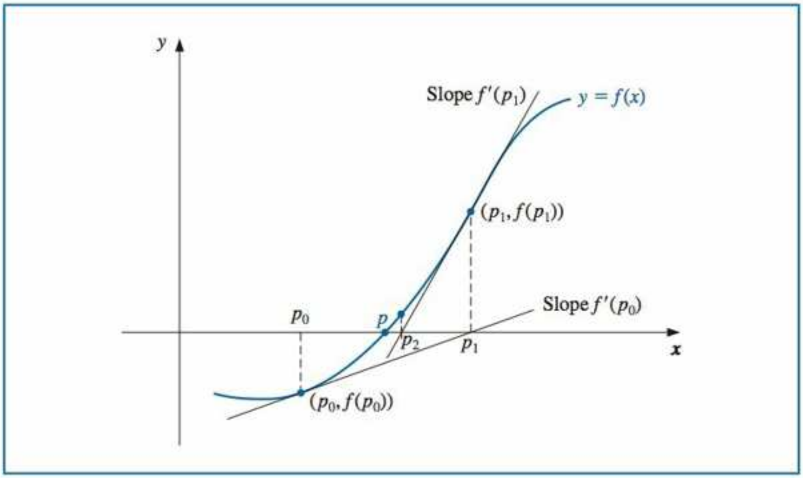
$$0 - f(p_0) = f'(p_0)(p_1 - p_0),$$

which implies that

$$p_1 = p_0 - \frac{f(p_0)}{f'(p_0)},$$

provided that $f'(p_0) \neq 0$. Subsequent approximations are found for p in a similar manner, as shown in Figure 2.6.

Figure 2.6



Newton's Method

The approximation p_{n+1} to a root of $f(x) = 0$ is computed from the approximation p_n using the equation

$$p_{n+1} = p_n - \frac{f(p_n)}{f'(p_n)},$$

provided that $f'(p_n) \neq 0$.

Example 1 Use Newton's method with $p_0 = 1$ to approximate the root of the equation $x^3 + 4x^2 - 10 = 0$.

Solution We will use MATLAB to find the first two iterations of Newton's method with $p_0 = 1$. We first define $f(x)$, $f'(x)$, and $p0$ with

Program NEWTON24 implements Newton's method.

```
f=inline('x^3+4*x^2-10','x')
fp = inline('3*x^2+8*x','x')
p0=1
```

The first iteration of Newton's method gives $p1 = 1.454545454545455$ using the command

```
p1=p0-f(p0)/fp(p0)
```

The second iteration is $p2 = 1.368900401069519$ using

```
p2=p1-f(p1)/fp(p1)
```

and so on. The process can be continued to generate the entries in Table 2.4. This table was generated using $p_0 = 1$, $TOL = 0.0005$, and $N_0 = 20$ in the program NEWTON24. Note that we have

$$|p - p_4| \approx 10^{-10}.$$

If we compare the convergence of this method with those applied to this problem previously, we can see that the number of iterations needed to solve the problem by Newton's method is less than the number needed for the Secant method. Recall that the Secant method required less than half the iterations needed for the Bisection method. ■

Table 2.4

n	p_n	$f(p_n)$
1	1.4545454545	1.5401953418
2	1.3689004011	0.0607196886
3	1.3652366002	0.0001087706
4	1.3652300134	0.0000000004

Convergence Using Newton's Method

Newton's method generally produces accurate results in just a few iterations. With the aid of Taylor polynomials we can see why this is true. Suppose p is the solution to $f(x) = 0$ and that f'' exists on an interval containing both p and the approximation p_n. Expanding f in the first Taylor polynomial at p_n gives

$$f(x) = f(p_n) + f'(p_n)(x - p_n) + \frac{f''(\xi)}{2}(x - p_n)^2,$$

and evaluating at $x = p$ gives

$$0 = f(p) = f(p_n) + f'(p_n)(p - p_n) + \frac{f''(\xi)}{2}(p - p_n)^2,$$

where ξ lies between p_n and p. Consequently, if $f'(p_n) \neq 0$, we have

$$p - p_n + \frac{f(p_n)}{f'(p_n)} = -\frac{f''(\xi)}{2f'(p_n)}(p - p_n)^2.$$

Since

$$p_{n+1} = p_n - \frac{f(p_n)}{f'(p_n)},$$

this implies that

$$p - p_{n+1} = p - p_n + \frac{f(p_n)}{f'(p_n)} = -\frac{f''(\xi)}{2f'(p_n)}(p - p_n)^2.$$

If a positive constant M exists with $|f''(x)| \leq M$ on an interval about p, and if p_n is within this interval, then

$$|p - p_{n+1}| \leq \frac{M}{2|f'(p_n)|}|p - p_n|^2.$$

The important feature of this inequality is that if f' is nonzero on the interval, then the error $|p - p_{n+1}|$ of the $(n+1)$st approximation is bounded by approximately the square of the error of the nth approximation, $|p - p_n|$. This implies that Newton's method has the tendency to approximately double the number of digits of accuracy with each successive approximation. Newton's method is not, however, infallible, as we will see later in this section.

Example 2 Find an approximation to the solution of the equation $x = 3^{-x}$ that is accurate to within 10^{-8}.

Solution A solution to $x = 3^{-x}$ corresponds to a solution of

$$0 = f(x) = x - 3^{-x}.$$

Since f is continuous with $f(0) = -1$ and $f(1) = \frac{2}{3}$, a solution of the equation lies in the interval (0, 1). We have chosen the initial approximation to be the midpoint of this interval, $p_0 = 0.5$. Succeeding approximations are generated by applying the formula

$$p_{n+1} = p_n - \frac{f(p_n)}{f'(p_n)} = p_n - \frac{p_n - 3^{-p_n}}{1 + 3^{-p_n}\ln 3}.$$

These approximations are listed in Table 2.5, together with differences between successive approximations. Since Newton's method tends to double the number of decimal places of accuracy with each iteration, it is reasonable to suspect that p_3 is correct at least to the places listed. ■

Table 2.5

n	p_n	$\lvert p_n - p_{n-1}\rvert$
0	0.500000000	
1	0.547329757	0.047329757
2	0.547808574	0.000478817
3	0.547808622	0.000000048

The success of Newton's method is predicated on the assumption that the derivative of f is nonzero at the approximations to the zero p. If f' is continuous, this means that the technique will be satisfactory provided that $f'(p) \neq 0$ and that a sufficiently accurate initial approximation is used. The condition $f'(p) \neq 0$ is not trivial; it is true precisely when p is a **simple zero**. A simple zero of a function f occurs at p if a function q exists with the property that, for $x \neq p$,

$$f(x) = (x - p)q(x), \qquad \text{where} \qquad \lim_{x \to p} q(x) \neq 0.$$

In general, a **zero of multiplicity** m of a function f occurs at p if a function q exists with the property that, for $x \neq p$,

$$f(x) = (x - p)^m q(x), \qquad \text{where} \qquad \lim_{x \to p} q(x) \neq 0.$$

So a simple zero is one that has multiplicity 1.

By taking consecutive derivatives and evaluating at p it can be shown that

- A function f with m derivatives at p has a zero of multiplicity m at p if and only if

$$0 = f(p) = f'(p) = \cdots = f^{(m-1)}(p), \text{ but } f^{(m)}(p) \neq 0.$$

When the zero is not simple, Newton's method might converge, but not with the speed we have seen in our previous examples.

Example 3 Let $f(x) = e^x - x - 1$. **(a)** Show that f has a zero of multiplicity 2 at $x = 0$. **(b)** Show that Newton's method with $p_0 = 1$ converges to this zero but not as rapidly as the convergence in Examples 1 and 2.

Table 2.6

n	p_n
0	1.0
1	0.58198
2	0.31906
3	0.16800
4	0.08635
5	0.04380
6	0.02206
7	0.01107
8	0.005545
9	2.7750×10^{-3}
10	1.3881×10^{-3}
11	6.9411×10^{-4}
12	3.4703×10^{-4}
13	1.7416×10^{-4}
14	8.8041×10^{-5}
15	4.2610×10^{-5}
16	1.9142×10^{-6}

Solution **(a)** We have

$$f(x) = e^x - x - 1, \qquad f'(x) = e^x - 1 \qquad \text{and} \quad f''(x) = e^x,$$

so

$$f(0) = e^0 - 0 - 1 = 0, \qquad f'(0) = e^0 - 1 = 0 \qquad \text{and} \quad f''(0) = e^0 = 1.$$

This implies that f has a zero of multiplicity 2 at $x = 0$.

(b) The first two terms generated by Newton's method applied to f with $p_0 = 1$ are

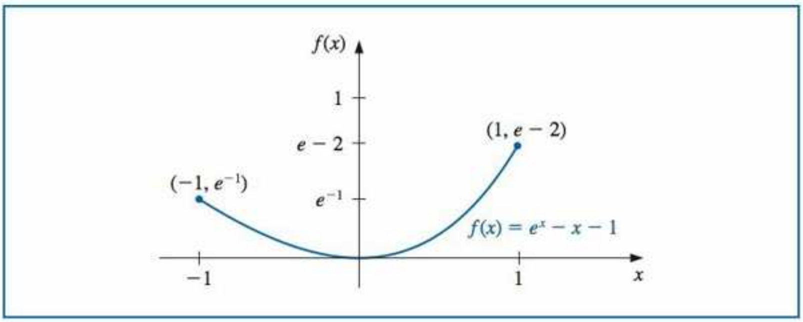
$$p_1 = p_0 - \frac{f(p_0)}{f'(p_0)} = 1 - \frac{e-2}{e-1} \approx 0.58198,$$

and

$$p_2 = p_1 - \frac{f(p_1)}{f'(p_1)} \approx 0.58198 - \frac{0.20760}{0.78957} \approx 0.31906.$$

The first eight terms of the sequence generated by Newton's method are shown in Table 2.6. The sequence is clearly converging to 0, but not as rapidly as the convergence in Examples 1 and 2. The graph of f is shown in Figure 2.7. ∎

Figure 2.7



One method for improving the convergence to a multiple root is considered in Exercise 8.

EXERCISE SET 2.4

1. Let $f(x) = x^2 - 6$ and $p_0 = 1$. Use Newton's method to find p_2.
2. Let $f(x) = -x^3 - \cos x$ and $p_0 = -1$. Use Newton's method to find p_2. Could $p_0 = 0$ be used for this problem?
3. Use Newton's method to find solutions accurate to within 10^{-4} for the following problems.
   a. $x^3 - 2x^2 - 5 = 0$, on $[1, 4]$
   b. $x^3 + 3x^2 - 1 = 0$, on $[-3, -2]$
   c. $x - \cos x = 0$, on $[0, \pi/2]$
   d. $x - 0.8 - 0.2 \sin x = 0$, on $[0, \pi/2]$
4. Use Newton's method to find solutions accurate to within 10^{-5} for the following problems.
   a. $2x \cos 2x - (x - 2)^2 = 0$, on $[2, 3]$ and $[3, 4]$
   b. $(x - 2)^2 - \ln x = 0$, on $[1, 2]$ and $[e, 4]$
   c. $e^x - 3x^2 = 0$, on $[0, 1]$ and $[3, 5]$
   d. $\sin x - e^{-x} = 0$, on $[0, 1]$, $[3, 4]$, and $[6, 7]$
5. Use Newton's method to find all four solutions of $4x \cos(2x) - (x - 2)^2 = 0$ in $[0, 8]$ accurate to within 10^{-5}.
6. Use Newton's method to find all solutions of $x^2 + 10 \cos x = 0$ accurate to within 10^{-5}.
7. Use Newton's method to approximate the solutions of the following equations to within 10^{-5} in the given intervals. In these problems, the convergence will be slower than normal because the zeros are not simple.
   a. $x^2 - 2xe^{-x} + e^{-2x} = 0$, on $[0, 1]$
   b. $\cos(x + \sqrt{2}) + x(x/2 + \sqrt{2}) = 0$, on $[-2, -1]$
   c. $x^3 - 3x^2(2^{-x}) + 3x(4^{-x}) + 8^{-x} = 0$, on $[0, 1]$
   d. $e^{6x} + 3(\ln 2)^2 e^{2x} - (\ln 8)e^{4x} - (\ln 2)^3$, on $[-1, 0]$
8. The numerical method defined by

$$p_n = p_{n-1} - \frac{f(p_{n-1})f'(p_{n-1})}{[f'(p_{n-1})]^2 - f(p_{n-1})f''(p_{n-1})},$$

   for $n = 1, 2, \ldots$, can be used instead of Newton's method for equations having multiple zeros. Repeat Exercise 7 using this method.
9. Use Newton's method to find an approximation to $\sqrt{3}$ correct to within 10^{-4}, and compare the results to those obtained in Exercise 9 of Sections 2.2 and 2.3.
10. Use Newton's method to find an approximation to $\sqrt[3]{25}$ correct to within 10^{-6}, and compare the results to those obtained in Exercise 10 of Sections 2.2 and 2.3.
11. Newton's method applied to the function $f(x) = x^2 - 2$ with a positive initial approximation p_0 converges to the only positive solution, $\sqrt{2}$.
   a. Show that Newton's method in this situation assumes the form that the Babylonians used to approximate $\sqrt{2}$:

$$p_{n+1} = \frac{1}{2} p_n + \frac{1}{p_n}.$$

   b. Use the sequence in (a) with $p_0 = 1$ to determine an approximation that is accurate to within 10^{-5}.
12. In Exercise 14 of Section 2.3, we found that for $f(x) = \tan \pi x - 6$, the Bisection method on $[0, 0.48]$ converges more quickly than the method of False Position with $p_0 = 0$ and $p_1 = 0.48$. Also, the Secant method with these values of p_0 and p_1 does not give convergence. Apply Newton's method to this problem with (a) $p_0 = 0$ and (b) $p_0 = 0.48$. (c) Explain the reason for any discrepancies.
13. Use Newton's method to determine the first positive solution to the equation $\tan x = x$, and explain why this problem can give difficulties.

14. Use Newton's method to solve the equation

$$0 = \frac{1}{2} + \frac{1}{4}x^2 - x\sin x - \frac{1}{2}\cos 2x, \quad \text{with } p_0 = \frac{\pi}{2}.$$

Iterate using Newton's method until an accuracy of 10^{-5} is obtained. Explain why the result seems unusual for Newton's method. Also, solve the equation with $p_0 = 5\pi$ and $p_0 = 10\pi$.

15. Player A will shut out (win by a score of 21–0) player B in a game of racquetball with probability

$$P = \frac{1+p}{2}\left(\frac{p}{1-p+p^2}\right)^{21},$$

where p denotes the probability that A will win any specific rally (independent of the server). (See [K,J], p. 267.) Determine, to within 10^{-3}, the minimal value of p that will ensure that A will shut out B in at least half the matches they play.

16. The function described by $f(x) = \ln(x^2+1) - e^{0.4x}\cos\pi x$ has an infinite number of zeros.

- **a.** Determine, within 10^{-6}, the only negative zero.
- **b.** Determine, within 10^{-6}, the four smallest positive zeros.
- **c.** Determine a reasonable initial approximation to find the nth smallest positive zero of f. [*Hint:* Sketch an approximate graph of f.]
- **d.** Use (c) to determine, within 10^{-6}, the 25th smallest positive zero of f.

17. The accumulated value of a savings account based on regular periodic payments can be determined from the *annuity due equation,*

$$A = \frac{P}{i}[(1+i)^n - 1].$$

In this equation, A is the amount in the account, P is the amount regularly deposited, and i is the rate of interest per period for the n deposit periods. An engineer would like to have a savings account valued at \$750,000 upon retirement in 20 years and can afford to put \$1500 per month toward this goal. What is the minimum interest rate at which this amount can be invested, assuming that the interest is compounded monthly?

18. Problems involving the amount of money required to pay off a mortgage over a fixed period of time involve the formula

$$A = \frac{P}{i}[1 - (1+i)^{-n}],$$

known as an *ordinary annuity equation.* In this equation, A is the amount of the mortgage, P is the amount of each payment, and i is the interest rate per period for the n payment periods. Suppose that a 30-year home mortgage in the amount of \$135,000 is needed and that the borrower can afford house payments of at most \$1000 per month. What is the maximum interest rate the borrower can afford to pay?

19. A drug administered to a patient produces a concentration in the blood stream given by $c(t) = Ate^{-t/3}$ milligrams per milliliter t hours after A units have been injected. The maximum safe concentration is 1 mg/ml.

- **a.** What amount should be injected to reach this maximum safe concentration and when does this maximum occur?
- **b.** An additional amount of this drug is to be administered to the patient after the concentration falls to 0.25 mg/ml. Determine, to the nearest minute, when this second injection should be given.
- **c.** Assuming that the concentration from consecutive injections is additive and that 75% of the amount originally injected is administered in the second injection, when is it time for the third injection?

20. Let $f(x) = 3^{3x+1} - 7 \cdot 5^{2x}$.

- **a.** Use the MATLAB function `fzero` to try to find all zeros of f.
- **b.** Plot $f(x)$ to find initial approximations to roots of f.
- **c.** Use Newton's method to find roots of f to within 10^{-16}.
- **d.** Find the exact solutions of $f(x) = 0$ algebraically.

2.5 Error Analysis and Accelerating Convergence

In the previous section we found that Newton's method generally converges very rapidly if a sufficiently accurate initial approximation has been found. This rapid speed of convergence is due to the fact that Newton's method produces *quadratically* convergent approximations.

Order of Convergence

Suppose that a method produces a sequence $\{p_n\}$ of approximations that converge to a number p.

- The sequence converges **linearly** if, for large values of n, a constant $0 < M$ exists with

$$|p - p_{n+1}| \le M|p - p_n|.$$

- The sequence converges **quadratically** if, for large values of n, a constant $0 < M$ exists with

$$|p - p_{n+1}| \le M|p - p_n|^2.$$

The constant M is called an **asymptotic error constant.**

The following illustrates the advantage of quadratic over linear convergence.

Illustration Suppose that $\{p_n\}_{n=0}^{\infty}$ is linearly convergent to 0 with

$$\lim_{n\to\infty} \frac{|p_{n+1}|}{|p_n|} = 0.5$$

and that $\{\tilde{p}_n\}_{n=0}^{\infty}$ is quadratically convergent to 0 with the same asymptotic error constant, $M = 0.5$, so

$$\lim_{n\to\infty} \frac{|\tilde{p}_{n+1}|}{|\tilde{p}_n|^2} = 0.5.$$

For simplicity we assume that for each n we have

$$\frac{|p_{n+1}|}{|p_n|} \approx 0.5 \quad \text{and} \quad \frac{|\tilde{p}_{n+1}|}{|\tilde{p}_n|^2} \approx 0.5.$$

For the linearly convergent scheme, this means that

$$|p_n - 0| = |p_n| \approx 0.5|p_{n-1}| \approx (0.5)^2|p_{n-2}| \approx \cdots \approx (0.5)^n|p_0|,$$

whereas the quadratically convergent procedure has

$$\begin{aligned} |\tilde{p}_n - 0| = |\tilde{p}_n| &\approx 0.5|\tilde{p}_{n-1}|^2 \approx (0.5)[0.5|\tilde{p}_{n-2}|^2]^2 = (0.5)^3|\tilde{p}_{n-2}|^4 \\ &\approx (0.5)^3[(0.5)|\tilde{p}_{n-3}|^2]^4 = (0.5)^7|\tilde{p}_{n-3}|^8 \\ &\approx \cdots \approx (0.5)^{2^n-1}|\tilde{p}_0|^{2^n}. \end{aligned}$$

Table 2.7 illustrates the relative speed of convergence of the sequences to 0 if $|p_0| = |\tilde{p}_0| = 1$.